STRONG MEASURE ZERO IN POLISH GROUPS

MICHAEL HRUŠÁK AND ONDŘEJ ZINDULKA

Abstract. The notion of strong measure zero is studied in the context of Polish groups. In particular, the extent to which the theorem of Galvin, Mycielski and Solovay holds in the context of an arbitrary Polish group is studied. Hausdorff measure and dimension is used to characterize strong measure zero. The products of strong measure zero sets are examined. Sharp measure zero, a notion stronger that strong measure zero, is shown to be related to meager-additive sets in the Cantor set and Polish groups by a theorem very similar to the theorem of Galvin, Mycielski and Solovay.

1. Introduction

All spaces and topological groups considered are separable and metrizable.

A natural extension of a definition due to Borel (1919) [6] asserts that a metric space X has *strong measure zero* (**Smz**) if for any sequence $\langle \varepsilon_n : n \in \omega \rangle$ of positive real numbers there is a cover $\{U_n : n \in \omega\}$ of X such that $\operatorname{diam} U_n \leqslant \varepsilon_n$ for all n.

In the same paper Borel conjectured that every strong measure zero set of reals is countable. This was shown to be independent of the usual axioms of set theory by Sierpiński (1928) [44] and Laver (1976) [28]. Later it was observed by Carlson [9] that the *Borel Conjecture* actually implies a formally stronger statement that all separable **Smz** metric spaces are countable.

We shall investigate the behaviour of strong measure zero sets in arbitrary Polish groups. In a sense we shall investigate the world, where the Borel conjecture fails, as most if not all of our results are trivial if the Borel Conjecture holds.

The subject of inquiry of this work starts with the theorem of Galvin, Mycielski, and Solovay [13, 14] who, confirming a conjecture of Prikry, proved that *a set $A \subseteq \mathbb{R}$ is of strong measure zero if and only if $A + M \neq \mathbb{R}$ for every meager set $M \subseteq \mathbb{R}$.*

Relatively recently Kysiak [26] and Fremlin [12], independently, showed that an analogous theorem is true for all locally compact metrizable groups (see also [50]). We present a proof of Kysiak and Fremlin's result based on [19] and consider the natural question as to how far the result can be extended. The fact that the theorem does not in general hold for all Polish groups was established in [19] and [50] and extended in [20]. This depends on further set-theoretic axioms, as the result obviously holds for all Polish groups assuming, e.g., the Borel Conjecture.

Cardinal invariants associated with strong measure zero sets on $\mathbb{R}$, ω^ω, and 2^ω have been studied rather extensively in recent decades [1, 15, 51, 8]. We review some of these, concentrating on the *uniformity* invariant of the σ-ideal $\mathbf{Smz}(\mathbb{G})$ of strong

2010 *Mathematics Subject Classification.* 03E17, 22B05, 22A10, 54E53, 54E52.

Key words and phrases. strong measure zero, Polish group, Galvin-Mycielski-Solovay theorem.

The first author gratefully acknowledges support from PAPIIT grant IN 100317. The second author was supported from European Regional Development Fund-Project "Center for Advanced Applied Science" (No. CZ.02.1.01/0.0/0.0/16_019/0000778).

measure zero subsets of a Polish group $\mathbb{G}$. A version of the Galvin-Mycielski-Solovay theorem links this study to the investigation of the so-called *transitive coefficient* $\mathsf{cov}^*(\mathcal{M})$ in Polish groups [1, 31, 11].

It was probably the aforementioned result of Prikry, Galvin, Mycielski and Solovay that inspired a few notions of smallness on the real line and Cantor set akin to strong measure zero. E.g., a set $S \subseteq \mathbb{R}$ is *strongly meager* if $S + N \neq \mathbb{R}$ for each Lebesgue null set N; it is *null-additive* if $S + N$ is Lebesgue null for each Lebesgue null set N; and it is *meager-additive* if $S + M$ is meager for each meager set M. These notions easily extend to other Polish groups.

We will study the latter notion, which is obviously a strengthening of strong measure zero. Since the early nineties, meager-additive sets in the Cantor set receive quite some attention. Let us single out the remarkable paper of Shelah [42] that provides a proof that each null-additive set in the Cantor set 2^ω is meager-additive and also the underlying combinatorial characterizations of null-additive and meager-additive sets in 2^ω (cf. 7.7 below), and Pawlikowski's paper [36] providing fine combinatorics and study of the so called transitive coefficients mentioned above that are actually cardinal invariants of strong measure zero, meager-additive and null-additive sets, and of course Bartoszyński's book [1]. However, all nontrivial results on meager-additive sets depended heavily on the combinatorial and group structure of 2^ω. In 2009 Weiss [47, 48] found a method that made the theory transferrable to the real line. Only very recently in [54, 52] it was noted that there is a description of meager-additive sets that resembles very much the Borel's definition of strong measure zero. Metric spaces having this property were termed to have *sharp measure zero*. This allowed for the theory of meager-additive sets to extend to other Polish groups. We provide some highlights of the rather new theory of sharp measure in metric spaces and meager-additive sets, and sharp measure zero on 2^ω and on Polish groups, including calculation of the uniformity number of sharp measure zero and meager-additive sets.

Our set-theoretic notation is standard and follows e.g. [25, 21]. In particular, the set of finite ordinals is identified with the set of non-negative integers and denoted interchangeably by ω and $\mathbb{N}$. In the same vein, the non-negative integers themselves are identified with the set of smaller non-negative integers, in particular $2 = \{0, 1\}$.

All spaces considered are separable and metrizable, often endowed with a compatible metric denoted d. We denote by $B(x, \varepsilon)$ the closed ball with radius ε centered at x, the corresponding open ball will be denoted by $B^\circ(x, \varepsilon)$.

The product spaces of the type A^ω for some finite or countable set A are considered with the *metric of least difference* defined by $d(f, g) = 2^{-|f \wedge g|}$, where $f \wedge g = f{\restriction}n$ for $n = \min\{k : f(k) \neq g(k)\}$. The clopen balls in the space A^ω are represented by nodes of the tree $A^{<\omega}$, given $s \in A^{<\omega}$, we let $\langle s \rangle = \{f \in A^\omega : s \subseteq f\}$. Given a subtree T of $A^{<\omega}$, we let $[T] = \{f \in A^\omega : \forall n \in \omega\ f{\restriction}n \in T\}$ be the (closed) set of branches of T. A metric space is *analytic* if it is a continuous image of ω^ω, and it is *Borel* (*absolutely G_δ*, resp.) if it is Borel (G_δ, resp.) in its completion.

A *Polish group* is a separable, completely metrizable topological group. A compatible metric d on a separable metrizable group $\mathbb{G}$ is *left-invariant* if $d(zx, zy) = d(x, y)$ for any $x, y, z \in \mathbb{G}$.

A separable group $\mathbb{G}$ is a CLI group if it admits a complete left-invariant compatible metric. Abelian and locally compact Polish groups are CLI, while, e.g., the group S_∞ of all permutations of ω is not.

A separable group $\mathbb{G}$ is a TSI group if it admits a (both-sided) invariant compatible metric. Not every Polish group admits an invariant metric, but if it is compact or abelian, then it does. Also, any invariant metric on a Polish group is complete.

2. Strong measure zero in Polish groups

The notion of strong measure zero is in general neither a topological nor a metric property, but a *uniform* property; in particular, a uniformly continuous image of a $\mathbf{Smz}$ set is $\mathbf{Smz}$, and if X uniformly embeds into Y, then any set $A \subseteq X$ that is not $\mathbf{Smz}$ in X is not $\mathbf{Smz}$ in Y either.

As all left-invariant (equiv right-invariant) metrics on a separable metrizable group are uniformly equivalent the notion of strong measure zero becomes seemingly "topological": a subset S of a topological group $\mathbb{G}$ is *Rothberger bounded* if for every sequence $\langle U_n : n \in \omega \rangle$ of neighbourhoods of $1_{\mathbb{G}}$ there is a sequence $\langle g_n : n \in \omega \rangle$ of elements of the group $\mathbb{G}$ such that the family $\langle g_n \cdot U_n : n \in \omega \rangle$ covers S. It follows [12] that a subset of a Polish group $\mathbb{G}$ is Rothberger bounded if and only if it is strong measure zero w.r.t. some (any) left-invariant metric on $\mathbb{G}$.

Many of the results stated here could be phrased in the language of uniformities and/or in terms of the property of being Rothberger bounded (see [12] for such treatment).

Whenever $\mathbb{G}$ is a Polish group, $\mathbf{Smz}(\mathbb{G})$ denotes the family of strong measure zero sets with respect to any left-invariant metric (i.e., the (left) Rothberger bounded sets as described above).

Of course, the choice of left-invariant over right-invariant is arbitrary, one being isomorphic to the other via the inverse map of the group in question. In fact, both the left Rothberger bounded and right Rothberger bounded set form a σ-ideal which is invariant under both left and right translations.

Proposition 2.1. $\mathbf{Smz}(\mathbb{G})$ *is a bi-invariant σ-ideal.*

Proof. To see that $\mathbf{Smz}(\mathbb{G})$ is a σ-ideal let $\{X_n : n \in \omega\} \subseteq \mathbf{Smz}(\mathbb{G})$ and a sequence $\{U_n : n \in \omega\}$ of open subsets of $\mathbb{G}$ be given. Let $\{I_n : n \in \omega\}$ be a partition of ω into infinite sets. As each X_n is of strong measure zero, there is a sequence $\{g_i : i \in I_n\} \subseteq \mathbb{G}$ such that $X_n \subseteq \bigcup_{i \in I_n} g_i \cdot U_i$. Then $\bigcup_{n\in\omega} X_n \subseteq \bigcup_{i\in\omega} g_i \cdot U_i$.

Now, let $X \in \mathbf{Smz}(\mathbb{G})$ and $g \in \mathbb{G}$ be given.

To see that $g \cdot X \in \mathbf{Smz}(\mathbb{G})$, note that if $\{U_n : n \in \omega\}$ is a sequence of open subsets of $\mathbb{G}$ and $\{g_i : i \in \omega\} \subseteq \mathbb{G}$ is such that $X \subseteq \bigcup_{n\in\omega} g_n \cdot U_n$, then $g \cdot X \subseteq \bigcup_{n\in\omega} g \cdot g_n \cdot U_n$.

To show that $X \cdot g \in \mathbf{Smz}(\mathbb{G})$, let $\{U_n : n \in \omega\}$ be a sequence of open subsets of $\mathbb{G}$. Consider the open sets $\{U_n \cdot g^{-1} : n \in \omega\}$. As $X \in \mathbf{Smz}(\mathbb{G})$ here is a sequence $\{g_n : n \in \omega\} \subseteq \mathbb{G}$ such that $X \subseteq \bigcup_{n\in\omega} g_n \cdot (U_n \cdot g^{-1})$. Then $X \cdot g \subseteq \bigcup_{n\in\omega} g_n \cdot U_n$. $\square$

Now, assuming Borel conjecture, or assuming that the group $\mathbb{G}$ has an invariant metric, the left and right Rothberger bounded sets coincide. This is not true in general, though:

Example 2.2. Assuming CH, there is a left Rothberger bounded subset of the group of permutations S^∞ of ω which is not right Rothberger bounded.

Proof. Denote by Ω the set of all finite partial injective functions from some $n \in \omega$ to ω. Enumerate all sequences of elements of S_∞ as $\{y_\alpha : \alpha < \omega_1\}$, and all increasing functions from ω to ω as $\{f_\alpha : \alpha < \omega_1\}$.

We shall recursively construct $\{g_\alpha : \alpha < \omega_1\} \subseteq S_\infty$ and $\{z_\alpha : \alpha < \omega_1\} \subseteq S_\infty^\omega$ so that

(1) $\forall \beta < \alpha < \omega_1\ \exists n \in \omega\ g_\alpha \restriction f_\beta(n) = z_\beta(n) \restriction f_\beta(n)$, while
(2) $\forall \beta < \alpha < \omega_1\ \forall n \in \omega\ g_\alpha^{-1} \restriction n+1 \neq y_\alpha^{-1}(n) \restriction n+1$, and
(3) $\forall s \in \Omega$ and $m_0 < m_1$ the first two elements of $\omega \setminus \operatorname{rng}(s)\ \exists a \in [\omega]^{m_1+1}$
   (a) $\forall n \in a\ s \subseteq z_\alpha(n) \restriction f_\alpha(n)$,
   (b) $\forall n \in a\ m_0 \in \operatorname{rng}(z_\alpha(n) \restriction f_\alpha(n))$,
   (c) $\forall n \in a\ m_1 \notin \operatorname{rng}(z_\alpha(n) \restriction f_\alpha(n))$, and
   (d) $\forall i \neq j \in a\ z_\alpha(i)^{-1}(m_0) \neq z_\alpha(j)^{-1}(m_0)$.

It should be clear, that if this can be accomplished then (1) guaranties that the set $X = \{g_\alpha : \alpha < \omega_1\}$ is of strong measure zero, while (2) makes sure that X^{-1} is not. The condition (3) is there for the construction not to prematurely terminate.

Assume that g_β, z_β for $\beta < \alpha$ have been constructed. First choose $z_\alpha \in S_\infty^\omega$ satisfying (3). Then enumerate $\alpha = \{\beta_i : i \in \omega\}$ and recursively find $\{n_i : i \in \omega\}$ so that $s_i = z_{\beta_i}(n_i) \restriction f_{\beta_i}(n_i)$ satisfy

(i) $s_i \subseteq s_{i+1}$,
(ii) if $m_i = \min(\omega \setminus \operatorname{rng}(s_i))$ then $m_i \in \operatorname{rng}(s_{i+1})$,
(iii) $\forall n \leqslant m_i\ \exists k_n \leqslant n\ k_n \in \operatorname{rng}(s_i)\ s_i^{-1}(k_n) \neq y_\alpha^{-1}(n)(k_n)$.

Then let $g_\alpha = \bigcup_{i\in\omega} s_i$. Then $g_\alpha \in S_\infty$ satisfying (1) by (i) and (ii), and (2) by (iii).

To construct the sequence $\langle s_n : n \in \omega \rangle$ start with $s_{-1} = \emptyset$. Having found s_i, let $m < k$ be the first two elements of $\omega \setminus \operatorname{rng}(s_i)$. By (3), there is $n_{i+1} \in \omega$ such that $s_{i+1} = z_{\beta_{i+1}}(n_{i+1}) \restriction f_{\beta_{i+1}}(n_{i+1})$ is such that $\{m, k\} \cap \operatorname{rng}(s_{i+1}) = \{m\}$, and $s_{i+1}^{-1}(n) \neq y_\alpha(n)^{-1}(n)$ for every $m \leqslant n < k$. $\square$

There is a close relation between strong measure zero and Geometric measure theory which shall be explored later on in the text, in section 5. The first result in this direction is due to Besicovitch [4, 5] who showed that a set X of reals has strong measure zero if and only if every uniformly continuous image of X has Hausdorff dimension 0.

Here we shall characterize strong measure sets in Polish groups as exactly the sets of *universal invariant submeasure zero*, a result due to J. Grebík.

It is a classical result of Haar [16] that every locally compact Polish group admits an (essentially unique) left-invariant, countably additive, outer regular Borel measure. In a similar vein, we shall prove here that every Polish group admits a non-trivial countably subadditive, outer regular, left-invariant diffuse submeasure, a result used in the next section.

Recall that a function $\mu : \mathcal{P}(\mathbb{G}) \to \mathbb{R}^+ \cup \{\infty\}$ is a submeasure if $\mu(\emptyset) = 0$, and $\mu(A \cup B) \leqslant \mu(A) + \mu(B)$ whenever A, B are subsets of $\mathbb{G}$. A submeasure μ on $\mathbb{G}$ is

- *σ-subadditive* if $\mu(\bigcup_{n\in\omega} A_n) \leqslant \sum_{n\in\omega} \mu(A_n)$, for any $\{A_n : n \in \omega\} \subseteq \mathcal{P}(\mathbb{G})$,
- *outer regular* if $\mu(A) = \inf\{\mu(U) :\ A \subseteq U, U$ open in $X\}$, for any $A \subseteq \mathbb{G}$,
- *left-invariant* if $\mu(A) = \mu(g \cdot A)$, for any $A \subseteq \mathbb{G}$ and $g \in \mathbb{G}$,
- *non-atomic* or *diffuse* if $\mu(\{x\}) = 0$ for every $x \in \mathbb{G}$, and
- *non-trivial* if $\mu(\mathbb{G}) > 0$.

Lemma 2.3. *In every Polish group* $\mathbb{G}$ *there is a decreasing local basis* $\{U_n : n \in \omega\}$ *of open sets at* $1_{\mathbb{G}}$ *such that for every* $m \in \omega$ *and* $\{a_n : n > m\} \subseteq \mathcal{P}(\mathbb{G})$ *such that* $|a_n| = n$ *for every* $n > m$, $U_m \not\subseteq \bigcup_{n>m} a_n \cdot U_n$.

Proof. Let d be a left invariant compatible metric on $\mathbb{G}$, and let e be a complete metric on $\mathbb{G}$. Recursively choose the open sets U_n, $n \in \omega$, together with finite sets $b_n \subseteq U_n$ of size $n+1$ so that:

In d: The points of b_n are $3 \operatorname{diam} U_{n+1}$ apart, and also $3 \operatorname{diam} U_{n+1}$ apart from the complement of U_n, while

In e: $\forall m < n \ \forall \{g_i : m < i < n\}$ with $g_i \in b_i$ $\operatorname{diam} \prod_{m<i<n} g_i \cdot U_n < \frac{1}{n}$.

To verify that the sequence $\{U_n : n \in \omega\}$ has the desired property assume that $m \in \omega$ and a sequence $\{a_n : n > m\} \subseteq \mathcal{P}(\mathbb{G})$ such that $|a_n| = n$ for every $n > m$ are given. Recursively choose $g_n \in b_n$ so that the set $\prod_{m<i<n} g_i \cdot U_n \cap a_n \cdot U_n = \emptyset$. Such g_n exists as d is left invariant, hence for every $g \in a_n$ the set $g \cdot U_n$ intersects at most one of the sets $\prod_{m<i<n} g_i \cdot h \cdot U_n$ for $h \in b_n$, and $|b_n| = |a_n| + 1$. The closures of the sets $\prod_{m<i<n} g_i \cdot U_n$, for $n > m$ form a decreasing sequence of sets of e-diameter converging to 0, hence by completeness of e their intersection is a singleton $x \in U_m$ which is not in $\bigcup_{n>m} a_n \cdot U_n$. □

Theorem 2.4 ([20])**.** *There is a non-trivial, left-invariant, outer regular,* σ*-sub-additive diffuse submeasure on every Polish group.*

Proof. Fix a sequence $\{U_n : n \in \omega\}$ as in Lemma 2.3 and define for $A \subseteq \mathbb{G}$:

$$\mu(A) = \inf \left\{ \sum_{i \in \omega} \frac{1}{n_i} : A \subseteq \bigcup_{i \in \omega} g_i \cdot U_{n_i} \right\}.$$

It is immediate from the definition that μ is a diffuse σ-additive, left invariant, outer regular submeasure on $\mathbb{G}$. To see that μ non-trivial it suffices to note that $\mu(U_m) = \frac{1}{m}$. To see that $\mu(U_m)$ is not less than $\frac{1}{m}$, note that by the key property of $\{U_n : n \in \omega\}$, if $U_m \subseteq \bigcup_{i \in \omega} g_i \cdot U_{n_i}$ then $\sum_{i \in \omega} \frac{1}{n_i} \geqslant \frac{1}{m}$. □

The promised characterization is the following:

Theorem 2.5 (J. Grebík, see [20])**.** *A subset* A *of a Polish group* $\mathbb{G}$ *is of left strong measure zero if and only if* $\mu(A) = 0$ *for every left-invariant, outer regular, countably additive diffuse submeasure on* $\mathbb{G}$.

Proof. Assume first that $X \in \mathsf{Smz}(\mathbb{G})$, let μ be a left-invariant, outer regular, countably additive diffuse submeasure on $\mathbb{G}$, and let $\varepsilon > 0$ be arbitrary. As μ is non-atomic and outer regular, there is a sequence $\{U_n : n \in \omega\}$ of neighborhoods of $1_{\mathbb{G}}$ such that $\sum_{n \in \omega} \mu(U_n) < \varepsilon$. Now, as $X \in \mathsf{Smz}(\mathbb{G})$, there is a sequence $\{g_n : n \in \omega\} \subseteq \mathbb{G}$ such that $X \subseteq \bigcup_{n \in \omega} g_n \cdot U_n$. By left invariance of μ, $\mu(X) \leqslant \sum_{n \in \omega} \mu(U_n) < \varepsilon$. Hence $\mu(X) = 0$.

On the other hand, assume that $X \subseteq \mathbb{G}$ has $\mu(X) = 0$ for every invariant, non-atomic, outer regular submeasure μ on $\mathbb{G}$, and let $\{V_n : n \in \omega\}$ be a sequence of open neighbourhoods of $1_{\mathbb{G}}$ in $\mathbb{G}$. Let $\{U_n : n \in \omega\}$ be a decreasing local basis as in Lemma 2.3, that is such that for every $m \in \omega$ and $\{a_n : n > m\}$ such that $|a_n| = n$ for every $n > m$, $U_m \not\subseteq \bigcup_{n>m} a_n \cdot U_n$, by passing on to a subsequence, we may assume that $U_n \subseteq V_n$ for every $n \in \omega$. Let $\{n_i : i \in \omega\} \subseteq \omega$ be such that $n_{i+1} > n_i$

for every $i \in \omega$, and let $\mathcal{W} = \{U_{n_{i+1}} : i \in \omega\}$, and $w(U_{n_{i+1}}) = \frac{1}{n_i}$. Then define a submeasure μ by putting for $A \subseteq \mathbb{G}$

$$\mu(A) = \inf \left\{ \sum_{i \in \omega} w(W_i) : A \subseteq \bigcup_{i \in \omega} g_i \cdot W_i \right\}$$

with each $W_i \in \mathcal{W}$ and $g_i \in \mathbb{G}$. This is again a left-invariant, σ-subadditive, non-atomic, outer regular submeasure, with $\mu(U_{n_{i+1}}) = \frac{1}{n_i}$. Hence $\mu(X) = 0$, in particular, there is a sequence $\{W_j : j \in \omega\}$ and a sequence $\{q_j : j \in \omega\}$ such that $X \subseteq \bigcup_{j \in \omega} g_i \cdot W_j$ and $\sum_{j \in \omega} w(W_j) < 1$. This means that every $U_{n_{i+1}}$ appears fewer that n_i-many times as one of the W_j, so there is permutation $\pi \in S_\infty$ such that $W_{\pi(n)} \subseteq V_n$ for every $n \in \omega$, hence $X \subseteq \bigcup_{n \in \omega} g_{\pi(n)} \cdot V_n$. Hence $X \in \mathsf{Smz}(\mathbb{G})$. □

In the general context of a metric space, Szpilrajn [45] proved that every **Smz** set X has universal measure zero, i.e. has measure zero for every finite diffuse Borel measure on X. It should be noted that unlike strong measure zero sets, uncountable universal measure zero sets exist in ZFC as shown by Sierpiński and Szpilrajn[43].

Proposition 2.6 (Szpilrajn [45])**.** *Strong measure zero sets in separable metric spaces have universal measure zero.*

Proof. Aiming towards contradiction, suppose that X is **Smz** yet there is a diffused Borel measure μ on X such that $\mu(X) = 1$. Define a function $f : (0, \infty) \to [0, 1]$ by

$$f(r) = \sup\{\mu(E) : \operatorname{diam} E \leqslant r\}.$$

We claim that $\lim_{r \to 0} f(r) = 0$. Otherwise there is $\varepsilon > 0$ and a sequence of sets E_n such that $\operatorname{diam} E_n \searrow 0$ and $\mu(E_n) \geqslant \varepsilon$. Let $E = \bigcap_{n \in \omega} \bigcup_{m \geqslant n} E_n$. Then clearly $\mu(E) \geqslant \varepsilon > 0$. In particular $E \neq \emptyset$, i.e., there is $I \in [\omega]^\omega$ such that $\bigcap_{n \in I} E_n \neq \emptyset$. Suppose without loss of generality that $I = \omega$. Since any two sets E_n, E_m have a common point, we have $\operatorname{diam}(\bigcup_{m \geqslant n} E_n) \leqslant 2 \operatorname{diam} E_n$. Therefore $\operatorname{diam} E \leqslant 2 \lim_{n \to 0} \operatorname{diam} E_n = 0$, which contradicts $\mu(E) > 0$. We proved that $\lim_{r \to 0} f(r) = 0$. Therefore there is, for each $n \in \omega$, $\varepsilon_n > 0$ such that $\sum_n f(\varepsilon_n) < 1$. Since X is **Smz**, there are sets U_n such that $\operatorname{diam} U_n < \varepsilon_n$ that cover X. It follows that

$$1 = \mu(X) \leqslant \sum_n \mu(U_n) \leqslant \sum_n f(\operatorname{diam} U_n) \leqslant \sum_n f(\varepsilon_n) < 1,$$

the desired contradiction. □

3. The Galvin-Mycielski-Solovay Theorem in Polish groups

In this section we study the Galvin-Mycielski-Solovay theorem in the context of an arbitrary Polish group $\mathbb{G}$. We denote by $\mathcal{M}(\mathbb{G})$, or simply by $\mathcal{M}$ if there is no danger of confusion, the ideal of meager subsets of $\mathbb{G}$. Much of this section exists thanks to the following simple yet crucial observation due to Prikry:

Proposition 3.1 (Prikry [38])**.** *Let $\mathbb{G}$ be a separable group, and let $S \subseteq \mathbb{G}$ be such that $S \cdot M \neq \mathbb{G}$ for all $M \in \mathcal{M}(\mathbb{G})$. Then $S \in \mathsf{Smz}(\mathbb{G})$.*

Proof. Let S be as above, and let $\{U_n : n \in \omega\}$ be a family of open neighbourhoods of 1 in $\mathbb{G}$. Let $\{g_n : n \in \omega\} \subseteq \mathbb{G}$ be such that $U = \bigcup_{n \in \omega} g_n \cdot U_n$ is dense open in $\mathbb{G}$. Then U^{-1} is dense open in $\mathbb{G}$, the inverse being a homeomorphism, so $M = \mathbb{G} \setminus U^{-1}$ is nowhere dense in $\mathbb{G}$. As $S \cdot M \neq \mathbb{G}$, there is $x \in \mathbb{G} \setminus S \cdot M$, that is $S \subseteq x \cdot U = \bigcup_{n \in \omega} x \cdot g_n \cdot U_n$. Hence, $S \in \mathsf{Smz}(\mathbb{G})$. □

As mentioned in the introduction, Galvin, Mycielski, and Solovay [13, 14] answered Prikry's question by showing that the reverse inclusion holds for $\mathbb{R}$. The same was recently proved for all locally compact groups by Kysiak [26] and Fremlin [12], independently. We shall present a proof of their theorem (the converse of Prikry's result for locally compact groups) here. Our proof follows [19].

Call a subset N of a topological group $\mathbb{G}$ *uniformly nowhere dense* if for every neighborhood U of 1 there is a neighborhood V of 1 such that for every $x \in \mathbb{G}$ there is a $g \in \mathbb{G}$ such that $g \cdot V \subseteq x \cdot U \setminus N$. A set $M \subseteq \mathbb{G}$ is *uniformly meager* if it can be written as a union of countably many uniformly nowhere dense sets. We denote the family of all uniformly meager subsets of $\mathbb{G}$ by $\mathcal{UM}(\mathbb{G})$ (or simply $\mathcal{UM}$). The following generalizes [14, Theorem 4].

Proposition 3.2 ([19])**.** *Let $\mathbb{G}$ be a Polish group which is either locally compact or* TSI*, and let $S \in \mathbf{Smz}(\mathbb{G})$. Then $S \cdot M \neq \mathbb{G}$ for all $M \in \mathcal{UM}(\mathbb{G})$.*

Proof. Assume first that $\mathbb{G}$ admits a invariant metric d. Recall that every invariant metric on a Polish group is complete. Let N be uniformly nowhere dense subset of $\mathbb{G}$. Note that for every $y \in \mathbb{G}$ and an open set $U \subseteq \mathbb{G}$, $y \cdot U \cdot N = U \cdot y \cdot N$, and $y \cdot N$ is uniformly nowhere dense. It follows that for every uniformly nowhere dense $N \subseteq \mathbb{G}$

$$\forall U \text{open } \exists V \text{open } \forall x, y \in \mathbb{G} \; \exists z \in \mathbb{G} \quad z \cdot V \subseteq x \cdot U \setminus V \cdot y \cdot N. \tag{1}$$

Now, fix a $\mathbf{Smz}$ set S and a uniformly meager set M written as the union of an increasing sequence $\langle N_n : n \in \omega \rangle$ of uniformly nowhere dense sets. Then there is a sequence $\langle U_n : n \in \omega \rangle$ of open subsets of $\mathbb{G}$, such that for every $n > 0$

$$\forall x, y \in \mathbb{G} \; \exists z \in \mathbb{G} \quad z \cdot U_n \subseteq x \cdot U_{n-1} \setminus U_n \cdot y \cdot N_n. \tag{2}$$

As S is $\mathbf{Smz}$, for every sequence $\{U_n : n \in \omega\}$ of open sets (of diameter converging to 0) there is a sequence $\{g_n : n \in \omega\} \subseteq \mathbb{G}$ such that each $s \in S$ is contained in infinitely many of the sets $g_n \cdot U_n$. Applying (2) recursively there is a sequence $\langle x_n : n \in \omega \rangle$ of elements of $\mathbb{G}$ such that for every $n \in \omega$

$$x_{n+1} \cdot g_{n+1} \cdot U_{n+1} \subseteq x_n \cdot g_n \cdot U_n \setminus (g_{n+1} \cdot U_{n+1} \cdot N_{n+1}).$$

The sequence $\langle x_n : n \in \omega \rangle$ is Cauchy, let x be its limit, i.e., $\{x\} = \bigcap_{n \in \omega} x_n \cdot g_n \cdot U_n$. Then $x \notin \bigcup_{n \in \omega} g_n \cdot U_n \cdot N_n \supseteq S \cdot M$, as the sequence $\langle N_n : n \in \omega \rangle$ is increasing and every element of S is contained in infinitely many of the $g_n \cdot U_n$ (for every $(s, m) \in S \times M$ there is an $n \in \omega$ such that $s \in g_n \cdot U_n$ and $m \in N_n$).

Now if $\mathbb{G}$ is locally compact, the proof proceeds along similar lines. Only (1) is replaced by the following lemma.

Lemma 3.3. *Let $\mathbb{G}$ be a locally compact Polish group equipped with a complete metric d, and let $U \subseteq \mathbb{G}$ be an open set with compact closure $C = \overline{U}$ and $P \subseteq \mathbb{G}$ be compact nowhere dense. Then*

$$\forall \varepsilon > 0 \; \exists \delta > 0 \; \forall x \in C \; \forall y \in K \; \exists z \in C \quad B(z, \delta) \subseteq B(x, \varepsilon) \setminus (B(y, \delta) \cdot P).$$

Proof. Fix $\varepsilon > 0$ and define $f : C \times K \to \mathbb{R}$ by

$$f(x, y) = \sup\{t : \exists z \in C \; B(z, t) \subseteq B^{\circ}(x, \varepsilon) \setminus y \cdot P)\}.$$

Then f is positive on $C \times K$ and attains its (positive) minimum.

To see that, consider, for each $z \in C$, the functions

$$g_z(x) = \underline{d}(z, X \setminus B^\circ(x, \varepsilon)), \quad x \in C$$
$$h_z(y) = \underline{d}(z, y \cdot P), \quad y \in K$$

and note that

$$f(x, y) = \sup_{z \in C} \min(g_z(x), h_z(y)). \tag{3}$$

Using compactness it is easy to see that, for each $z \in C$, the function h_z is lower semicontinuous and that while g_z does not have to be, it has the following lower-semicontinuity property: if $x_n \to x$ and $g_z(x_n) \to 0$, then $g_z(x) = 0$.

Now suppose that there are $(x_n, y_n) \in C \times K$ such that $f(x_n, y_n) \to 0$. Since C, K are compact, passing to subsequences we may assume $(x_n, y_n) \to (x, y) \in C \times K$. Use (3) and the semicontinuity properties of g_z and h_z to conclude that since $f(x_n, y_n) \to 0$, for any z either $g_z(x_n) \to 0$ and then $g_z(x) = 0$, or else $h_z(y_n) \to 0$ and then $h_z(y) = 0$. Use (3) again to conclude that $f(x, y) = 0$, the desired contradiction proving that there is $\eta > 0$ such that $f(x, y) > \eta$ for all x, y. It follows that

$$\forall x \in C\ \forall y \in K\ \exists z \in C \quad B(z, \eta) \subseteq B(x, \varepsilon)\ \wedge B(z, \eta) \cap y \cdot P = \emptyset.$$

The latter of course yields $B(z, \frac{\eta}{2}) \cap B(y \cdot P, \frac{\eta}{2}) = \emptyset$. On the other hand, there is $\xi > 0$ such that

$$\forall y \in K \quad B(y, \xi) \cdot P \subseteq B(y \cdot P, \tfrac{\eta}{2}).$$

It follows that $B(z, \frac{\eta}{2}) \cap B(y, \xi) \cdot P = \emptyset$. Thus letting $\delta = \min\{\frac{\eta}{2}, \xi\}$ yields the lemma. □

To conclude, write $\mathbb{G}$ as the union of an increasing sequence of open sets with compact closures K_n, and write a meager set M as the union of an increasing sequence of compact nowhere dense sets P_n. Choose $x_0 \in \mathbb{G}$ and $\varepsilon_0 > 0$ such that $B(x_0, \varepsilon_0)$ is compact. Let $C = B(x_0, \varepsilon_0)$. By the above lemma there is a sequence $\langle \varepsilon_n : n \in \omega \rangle \in (0, \infty)^\omega$ such that for every $n > 0$

$$\forall x \in C\ \forall y \in K_n\ \exists z \in C \quad B(z, \varepsilon_n) \subseteq B(x, \varepsilon_{n-1}) \setminus B(y, \varepsilon_n) \cap K_n) \cdot P_n. \tag{4}$$

We may of course suppose that $\varepsilon_n \to 0$. Since S is **Smz**, there is a cover $\{E_n\}$ of S such that $\operatorname{diam} E_n < \varepsilon_n$ for all n such that each point of S is covered by infinitely many E_n's).

For each n there is y such that $E_n \subseteq B(y, \varepsilon_n)$. Therefore, using repeatedly (4), there is a sequence $\langle x_n : n \in \omega \rangle$ in C such that for all $n \in \omega$

$$B(x_{n+1}, \varepsilon_{n+1}) \subseteq B(x_n, \varepsilon_n) \setminus (E_{n+1} \cap K_{n+1}) \cdot P_{n+1}.$$

Let x be the unique point of $\bigcap_{n \in \omega} B(x_n, \varepsilon_n)$ (there is one, since $B(x_0, \varepsilon_0)$ is compact and is unique as $\varepsilon_n \to 0$). Then $x \notin \bigcup_{n \in \omega} (E_n \cap K_n) \cdot P_n$.

Thus, to prove that x is not covered by $S \cdot M$ it suffices to show that $S \times M \subseteq \bigcup_{n \in \omega} (E_n \cap K_n) \times P_n$. Let $(s, m) \in S \times M$. There is k such that $(s, m) \in K_k \times P_k$. Since there are infinitely many n such that $s \in E_n$, there is $n \geqslant k$ such that $s \in E_n$, hence $s \in E_n \cap K_k \subseteq E_n \cap K_n$. Also $m \in P_k \subseteq P_n$. Therefore $(s, m) \in (E_n \cap K_n) \times P_n$. The desired inclusion is proved. □

The Galvin-Mycielski-Solovay/Fremlin/Kysiak result follows from the fact that in a locally compact group every meager set is uniformly meager:

Proposition 3.4 ([19])**.** *A Polish group $\mathbb{G}$ is locally compact if and only if $\mathcal{M}(\mathbb{G}) = \mathcal{UM}(\mathbb{G})$.*

Proof. We shall see first that $\mathcal{M}(X) = \mathcal{UM}(X)$ *for every locally compact metric space X.*

To that end it suffices to see that every nowhere dense subset of a compact space is, in fact, uniformly nowhere dense. Let N be a nowhere dense subset of a compact space X, and let $\varepsilon > 0$. Let F be a finite subset of X such that $Z = \bigcup_{x\in F} B(x, \frac{\varepsilon}{2})$. For every $x \in F$ let $y_x \in B(x, \frac{\varepsilon}{2})$ and $\delta_x > 0$ be such that $B(y_x, \delta_x) \subseteq B(x, \frac{\varepsilon}{2}) \setminus N$. Then $\delta = \min\{\delta_x : x \in F\}$ works as $B(x, \frac{\varepsilon}{2}) \subseteq B(z, \varepsilon)$ whenever $z \in B(x, \frac{\varepsilon}{2})$.

On the other hand, $\mathcal{M}(X) \neq \mathcal{UM}(X)$ *for every nowhere locally compact complete metric space X.*

To see this let X be nowhere locally compact with a complete metric d. Then for every U with non-empty interior there is an $\varepsilon^U > 0$ and a pairwise disjoint family $\{V^U_k : k \in \omega\}$ of open balls of radius ε^U contained in U.

We shall construct a nowhere dense set N which is not uniformly meager. To do that we recursively construct a family $\{U_s : s \in \omega^{<\omega}\}$ of non-empty regular closed[1] sets so that

(1) $\operatorname{diam} U_s \leqslant 2^{-|s|}$ for every $s \in \omega^{<\omega}$,
(2) $\bigcup_{n\in\omega} U_{s^\frown n} \subseteq U_s$ for every $s \in \omega^{<\omega}$,
(3) $U_{s^\frown n} \cap U_{s^\frown m} = \emptyset$ for every $s \in \omega^{<\omega}$ and any two distinct $m, n \in \omega$,
(4) $\operatorname{int}(U_s \setminus \bigcup_{n\in\omega} U_{s^\frown n}) \neq \emptyset$ for every $s \in \omega^{<\omega}$,
(5) for all $s \in \omega^{<\omega}$ and $k \in \omega$ and $x \in V^{U_s}_k$ there is a $n \in \omega$ such that $U_{s^\frown n} \subseteq B(x, 2^{-k})$.

To do this is straightforward.

Having constructed such a family, let $N = \bigcap_{j\in\omega} \bigcup_{|s|=j} U_s$. This is the required set:

It is nowhere dense as a non-empty open set U is either disjoint from N, or contains U_s for some $s \in \omega^{<\omega}$. Then, however, $\emptyset \neq \operatorname{int}(U_s \setminus \bigcup_{n\in\omega} U_{s^\frown n}) \subseteq U \setminus N$ by the property (4) above.

Now we will prove that N is not uniformly meager in X. The set N is naturally homeomorphic to ω^ω (see properties (1)–(3) above), hence satisfies the Baire Category Theorem. Aiming toward a contradiction assume that $N \subseteq \bigcup_{l\in\omega} N_l$, where each N_l is a closed uniformly nowhere dense subset of X. By the Baire Category Theorem applied to N there is an $s \in \omega^{<\omega}$ and an $l \in \omega$ such that $U_s \cap N \subseteq N_l$, hence $U_s \cap N$ is uniformly nowhere dense. So, there is a $\delta > 0$ as in the definition of uniformly nowhere dense corresponding to ε^{U_s}. Consider $V^{U_s}_k$, for $2^{-k} < \delta$. Then, on the one hand there is an $x \in V^{U_s}_k$ such that $B(x, 2^{-k}) \subseteq V^{U_s}_k \setminus N$, and on the other hand, there is (see property (5) above) an $n \in \omega$ such that $\emptyset \neq N \cap U_{s^\frown n} \subseteq B(x, 2^{-k})$, which is a contradiction.

The result follows as every Polish group is either locally compact or nowhere locally compact. □

And finally:

Theorem 3.5 (Fremlin [12], Kysiak [26])**.** *Let $\mathbb{G}$ be a locally compact Polish group. A set $A \subseteq \mathbb{G}$ is of strong measure zero if and only if $A \cdot M \neq \mathbb{G}$ for every meager set $M \subseteq \mathbb{G}$*

[1]Recall that a set U is *regular closed* if U is the closure of the interior of U.

Proof. The theorem follows directly from Propositions 3.1, 3.2 and 3.4. □

Next we shall discuss the possibility of extending the Galvin-Mycielski-Solovay theorem to a larger class of Polish groups. First, one needs to realize that assuming the Borel conjecture, the theorem holds trivially for every Polish group $\mathbb{G}$, as strong measure zero sets in all Polish groups are exactly the countable subsets ([9]), hence $S \cdot M$ is meager for every strong measure zero set S and meager set M, hence $S \cdot M \neq \mathbb{G}$.

On the other hand, it was shown in [19], that the Galvin-Mycielski-Solovay theorem fails for the Baer-Specker group $\mathbb{Z}^\omega$, assuming $\mathsf{cov}(\mathcal{M}) = \mathfrak{c}$. We conjecture that assuming a strong failure of the Borel conjecture the locally compact Polish groups are exactly the ones for which the theorem holds.

Conjecture 3.6 (CH)**.** *The Galvin-Mycielski-Solovay theorem holds in a Polish group $\mathbb{G}$ if and only if $\mathbb{G}$ is locally compact.*

The Continuum Hypothesis is optimal in the sense that under CH the Galvin-Mycielski-Solovay Theorem fails for as many Polish groups as possible follows from the logical complexity of the problem. The statement $\mathbb{G}$ *satisfies the Galvin-Mycielski-Solovay Theorem* is a Π^2_1-statement with $\mathbb{G}$ as a parameter, and hence is decided by the Ω-logic under the Continuum hypothesis. Moreover, if the statement is true assuming CH it is true in ZFC (see e.g. [27]).

We shall verify (following [20]) that the conjecture is true for Abelian Polish groups, in fact, it is true for all groups with a complete (both-sided)-invariant metric, and also for closed subgroups of the permutation group S_∞. Whether it is true in general remains open.

There is a, perhaps an even more interesting, stronger ZFC conjecture on the structure of Polish groups. The following concept was introduced in [19] and the term coined in [20]: A nonempty subset C of a Polish group $\mathbb{G}$ is said to be *anti-GMS* if it is nowhere dense and for every sequence $\{U_n : n \in \omega\}$ of open neighborhoods of 1 there is a sequence $\{g_n : n \in \omega\}$ of elements of $\mathbb{G}$ such that for every $g \in \mathbb{G}$, the set $g \cdot \bigcup_{n\in\omega} g_n \cdot U_n$ is dense in C.

The reason for introducing anti-GMS sets is the following:

Proposition 3.7 ([19])**.** *Assuming $\mathsf{cov}(\mathcal{M}) = \mathfrak{c}$, if $C \subseteq \mathbb{G}$ is anti-GMS, then there is a strong measure zero set S such that $S \cdot C = \mathbb{G}$.*

Proof. Enumerate $\mathbb{G} = \{g_\alpha : \alpha < \mathfrak{c}\}$ and enumerate all sequences of open sets in $\mathbb{G}$ as $\{\langle U^\alpha_n : n \in \omega\rangle : \alpha < \mathfrak{c}\}$. Let $C \subseteq \mathbb{G}$ be anti-GMS, and for every $\alpha < \mathfrak{c}$ let $\langle g^\alpha_n : n \in \omega\rangle$ be such that for all $g \in \mathbb{G}$, $(g \cdot \bigcup_{n\in\omega} g^\alpha_n \cdot U^\alpha_n) \cap M$ is comeager in M. Let $U_\alpha = \bigcup_{n\in\omega} g^\alpha_n \cdot U^\alpha_n$.

As $\mathsf{cov}(\mathcal{M}) = \mathfrak{c}$, the intersection of fewer than $\mathfrak{c}$ relatively dense open subsets of M is not empty. In particular, for every $\alpha < \mathfrak{c}$, there is an

$$m_\alpha \in M \cap (g^{-1}_\alpha \cdot \bigcap_{\beta\leqslant\alpha} U_\beta).$$

There is then an $x_\alpha \in \bigcap_{\beta\leqslant\alpha} U_\beta$ such that $m_\alpha = g^{-1}_\alpha \cdot x_\alpha$. That is $g_\alpha = x_\alpha \cdot m^{-1}_\alpha$.

Let $X = \{x_\alpha : \alpha < \mathfrak{c}\}$. Then $\mathbb{G} = X \cdot M^{-1}$. Let us see that $X \in \mathbf{Smz}(\mathbb{G})$: Given a sequence $\{U_n : n \in \omega\}$ of neighbourhoods of $1_\mathbb{G}$, consider the subsequence $\{U_{2n} : n \in \omega\}$. It is listed as $\langle U^\alpha_n : n \in \omega\rangle$ for some $\alpha < \mathfrak{c}$. By the construction, every $x_\gamma \in U_\alpha$ for $\gamma \geqslant \alpha$. On the other hand, $X \setminus U_\alpha \subseteq \{x_\beta : \beta < \alpha\}$, hence has size less than $\mathsf{cov}(\mathcal{M}) = \mathfrak{c}$. Committing the sin of forward reference, by Theorem 4.2(i), $X \setminus U_\alpha$ is a $\mathbf{Smz}$ set, hence can be covered by the sets $\{U_{2n+1} : n \in \omega\}$. Hence, X has strong measure zero. $\square$

The anti-GMS sets are our only tool for disproving the Galvin-Mycielski-Solovay Theorem in non-locally compact groups. Hence the *Strong Conjecture* is:

Conjecture 3.8. *Exactly one of the following holds for any Polish group $\mathbb{G}$: Either $\mathbb{G}$ is locally compact, or it contains an anti-GMS set.*

It is immediate from Proposition 3.7 that the Strong conjecture, indeed, solves the GMS conjecture stated above. It is, in fact the Strong conjecture we have verified for the aforementioned classes of groups:

Theorem 3.9 ([20]). *Let $\mathbb{G}$ be a non-locally compact* TSI *Polish group. Then $\mathbb{G}$ contains an anti-GMS set.*

Proof. Let μ be a non-trivial left-invariant countably subadditive, diffuse outer regular submeasure given by Theorem 2.4.

Claim 3.10. *There is a nowhere dense set $C \subseteq \mathbb{G}$ such that for every open set O intersecting C there is an open set U and $\{g_m : m \in \omega\} \subseteq \mathbb{G}$ such that for every $m \in \omega$ $g_m \cdot U \subseteq O$ and $\lim_{m\in\omega} \mu(g_m \cdot U \setminus C) = 0$.*

Proof. To construct the set C let $\{B_n : n \in \omega\}$ be a basis for the topology of $\mathbb{G}$. Recursively construct an increasing sequence of open sets $\{W_k : k \in \omega\}$ and a sequence $\{A_k : k \in \omega\}$ of countable sets of pairs of the form $\langle U, \varepsilon\rangle$, where U is an open subset of $\mathbb{G}$ and $\varepsilon > 0$ such that

(1) $B_k \cap W_{k+1} \neq \emptyset$,
(2) $\forall g \in \mathbb{G}\ |\{\langle U, \varepsilon\rangle \in A_k : g \in U\}| \leqslant k$,
(3) $\forall \langle U, \varepsilon\rangle \in A_k\ \mu(\overline{W_{k+1}} \cap U \setminus \overline{W}_k) < \frac{\varepsilon}{2^k}$, and if k is the least such that $\langle U, \varepsilon\rangle \in A_k$ then $\overline{W}_k \cap U = \emptyset$, and
(4) (a) either $B_k \subseteq W_{k+1}$,
  (b) or, there is an open neighborhood V of $1_\mathbb{G}$ and there are distinct $\{g_i : i \in \omega\} \subseteq \mathbb{G}$ and $\{\varepsilon_i : i \in \omega\}$ such that for all $i \in \omega$ $g_i \cdot V \subseteq B_n$, $\langle g_i \cdot V, \varepsilon_i\rangle \in A_k$ and $\lim_{i\in\omega} \varepsilon_i = 0$.

To carry out the construction, put first $W_0 = A_0 = \emptyset$. Having constructed W_k and A_k, see first whether $B_k \subseteq \overline{W}_k$. If so, let $A_{k+1} = A_k$ and let $W_{k+1} = W_k \cup B_k$. If $B_k \not\subseteq \overline{W}_k$, let U_0, U_1 be disjoint open subsets of $B_k \setminus \overline{W}_k$. By (2) there is an open set $U_2 \subseteq U_0$ contained or disjoint from every U appearing in A_k, by non-atomicity of μ we may require $\mu(U_2)$ to be so small that $W_{k+1} = W_k \cup U_2$ satisfies (3) for all $\langle U, \varepsilon\rangle \in A_k$ such that $U_2 \subseteq U$. Finally, as U_1 is not compact, there are infinitely many balls of the same diameter (i.e. translates of the same open set) with disjoint closures contained in U_1, add them to A_{k+1} paired with some real numbers converging to 0. It is clear that (1)-(4) are satisfied.

Now, let $C = \mathbb{G} \setminus \bigcup_{k\in\omega} W_k$. Then C is a closed nowhere dense set by (1). By (3), $\mu(U \setminus C) < \varepsilon$ for every $\langle U, \varepsilon\rangle \in \bigcup_{k\in\omega} A_k$, and if $B_k \cap C \neq \emptyset$ then by (4) B_k contains infinitely many sets with the required properties. $\square$

The set C from the claim is anti-GMS. In order to verify this let a sequence $\{U_m : m \in \omega\}$ of open neighborhoods of $1_\mathbb{G}$ be given, without loss of generality of diameter shrinking to 0. Let again $\{B_n : n \in \omega\}$ be a basis for the topology of $\mathbb{G}$, and let $\{A_n : n \in \omega\}$ be a partition of ω into infinite sets. For every B_n such that $B_n \cap C \neq \emptyset$ let W_n be an open neighborhood of $1_{\mathbb{G}}$ such that there are distinct $\{g_i : i \in \omega\} \subseteq \mathbb{G}$ such that $g_i \cdot W_n \subseteq B_n$ and $\lim_{i\in\omega} \mu(g_i \cdot W_n \setminus C) = 0$.

Now, let V_n be an open neighborhood of $1_{\mathbb{G}}$ such that $V_n \cdot U_j \subseteq W_n$ for all but finitely many $j \in A_n$, and let $g_j, h_j \in \mathbb{G}$ be such that $h_j \cdot W_n \subseteq B_n$, $\mu(h_j \cdot W_n \setminus C) < \mu(U_j)$, and $G = \bigcup_{j\in A_n} h_j \cdot V_n \cdot g_j^{-1}$ (here is where we use the invariance of the metric). The sequence $\{g_i : i \in \omega\} \subseteq \mathbb{G}$ witnesses (for the sequence $\{U_i : i \in \omega\}$ that C is anti-GMS. Indeed, of $g \in \mathbb{G}$ and $B_n \cap C \neq \emptyset$ then there is a $j \in A_n$ such that $g \in h_j \cdot V_n \cdot g_j^{-1}$, i.e. $g \cdot g_j \in h_j \cdot V_n$, hence

$$g \cdot g_j \cdot U_j \subseteq h_j \cdot V_n \cdot U_j \subseteq h_j \cdot W_n \subseteq B_n.$$

As $\mu(h_j \cdot W_n \setminus C) < \mu(U_j)$, we get $C \cap B_n \cap g \cdot g_j \cdot U_j \neq \emptyset$, as required. □

Corollary 3.11. *The strong conjecture is true for Abelian groups.*

We do not know, whether the strong conjecture is true for all Polish groups, but we can confirm it for another important class of groups – the automorphism groups of countable structures:

Theorem 3.12 ([20])**.** *Let $\mathbb{G}$ be a non-locally compact closed subgroup of S_∞. Then $\mathbb{G}$ contains an anti-GMS set.*

Proof. As $\mathbb{G}$ is not locally compact, there is an infinite $A \subseteq \omega$ such that for all $n \in A$ there are infinitely many $m \in \omega$ for which there are $g \in G$ such that $g{\restriction}n = 1_{\mathbb{G}}{\restriction}n$ and $g(n) = m$. Given $n, m \in A$, $n < m$ define

$$R_{n,m} = \{(g,h) \in \mathbb{G} \times \mathbb{G} : g(n) \notin \operatorname{rng}(h{\restriction}m) \text{ and } h(n) \notin \operatorname{rng}(g{\restriction}m)\}.$$

Note that the relation is left-invariant. Let $B = \{u \in \omega^{<\omega} : U$ is one-to-one and $dom(u) \in A$ and $u \subseteq g$ for some $g \in \mathbb{G}\}$.

Claim 3.13. *For every $n < m \in A$ and $u \in B$ such that $|u| = n$ there are $g, h \in \mathbb{G}$ both extending u such that $(g,h) \in R_{n,m}$.*

Proof. By the left-invariance of $R_{n,m}$, we may assume that u is the identity on $n = dom(u)$. Let $\{g_k : k \in \omega\} \subseteq \mathbb{G}$ be such that $g_k{\restriction}n = u$ and $\{g_k(n)) : k \in \omega\}$ are pairwise distinct, by further shrinking we may assume that for every $l \in [n, m)$ either $\{g_k(n)) : k \in \omega\}$ are pairwise distinct or all equal. In particular, for every $k_0 \in \omega$ the set $\{k_1 \in \omega : g_{k_0}(n) \in \operatorname{rng}(g_{k_1}{\restriction}[n,m)\}$ is finite, hence $(g_0, g_k) \in R_{n,m}$ for almost all $k \in \omega$. □

Given a subset H of B let $T(H) = \{u \in B : \forall k \in \omega\ u{\restriction}k \notin H\}$. Construct $D \subseteq B$ such that

(1) $\forall u \in B\ \exists v \in D\ u \subseteq v$,
(2) $\forall v, v' \in D\ \operatorname{rng}(v) \subseteq \operatorname{rng}(v')$ or $\operatorname{rng}(v') \subseteq \operatorname{rng}(v)$, and
(3) $\forall u \in T(D)\ \forall m \in \omega\ \exists v \in T(D)\ m \in \operatorname{rng}(v)$.

It is easy to construct such a set using a simple bookkeeping argument. Having done so, let $C = [T(D)]$.

To see that C is anti-GMS, consider an infinite set $Z \subseteq A$. By the Claim there is a sequence $\{g_n : n \in Z\} \subseteq \mathbb{G}$ such that

$$\forall u \in B\ \exists n_o < n_1 \in Z\ u \subseteq g_{n_0} \cap g_{n_1} \text{ and } (g_{n_o}, g_{n_1}) \in R_{|u|,n_0}.$$

To finish the proof it suffices to show that $g \cdot \bigcup_{n\in Z}\langle g_n{\restriction}n\rangle \cap C$ is dense in C for every $g \in \mathbb{G}$. To that end fix $g \in \mathbb{G}$ and $v \in B$ such that $\langle v\rangle \cap C \neq \emptyset$, and let $k = dom(v)$

and $u = g^{-1} \cdot v$. Choose $n_0 < n_1 \in Z$ such that $u \subseteq g_{n_0} \cap g_{n_1}$ and $(g_{n_o}, g_{n_1}) \in R_{k,n_1}$. Then either $\langle g \cdot g_{n_0} \restriction n_0 \rangle \cap C \neq \emptyset$ or $\langle g \cdot G_{n_1} \restriction n_1 \rangle \cap C \neq \emptyset$: If not, then there are $s_0, s_1 \in D$ such that $s_0 \subseteq g \cdot g_{n_0} \restriction n_0$ and $s_1 \subseteq g \cdot g_{n_1} \restriction n_1$. Neither $s_0 \subseteq v$ nor $s_1 \subseteq v$ as $C \cap \langle v \rangle \neq \emptyset$, so $g \cdot g_{n_0}(k) \in \mathrm{rng}(s_1)$ and $g \cdot g_{n_1}(k) \in \mathrm{rng}(s_0)$. This, however, contradicts the assumption that $\mathrm{rng}(s_0) \subseteq \mathrm{rng}(s_1)$ or $\mathrm{rng}(s_1) \subseteq \mathrm{rng}(s_0)$. □

4. Cardinal invariants of **Smz** in Polish groups

Given an ideal $\mathcal{I}$ of subsets of a set X the following are the standard cardinal invariants associated with $\mathcal{I}$:

$$\begin{aligned}
\mathsf{non}(\mathcal{I}) &= \min\{|Y| : Y \subseteq X \wedge Y \notin \mathcal{I}\}, \\
\mathsf{add}(\mathcal{I}) &= \min\{|\mathcal{A}| : \mathcal{A} \subseteq \mathcal{I} \wedge \bigcup \mathcal{A} \notin \mathcal{I}\}, \\
\mathsf{cov}(\mathcal{I}) &= \min\{|\mathcal{A}| : \mathcal{A} \subseteq \mathcal{I} \wedge \bigcup \mathcal{A} = X\}, \\
\mathsf{cof}(\mathcal{I}) &= \min\{|\mathcal{A}| : \mathcal{A} \subseteq \mathcal{I} \wedge (\forall I \in \mathcal{I})(\exists A \in \mathcal{A})(I \subseteq A)\}.
\end{aligned}$$

We denote by $\mathcal{M}, \mathcal{N}$ the ideals of meager and Lebesgue null subsets of 2^ω, respectively. For $f, g \in \omega^\omega$, we say that $f \leqslant^* g$ if $f(n) \leqslant g(n)$ for all but finitely many $n \in \omega$ (the order of *eventual dominance*). A family $F \subseteq \omega^\omega$ is *bounded* if there is an $h \in \omega^\omega$ such that $f \leqslant^* h$ for all $f \in F$; and F is *dominating* if for any $g \in \omega^\omega$ there is $f \in F$ such that $g \leqslant^* f$. The cardinal invariants related to eventual dominance are $\mathfrak{b}$ (the minimal cardinality of an unbounded family) and $\mathfrak{d}$ (the minimal cardinality of a dominating family).

We shall briefly review the results (not necessarily in the chronological order) concerning other cardinal invariants of **Smz**($\mathbb{G}$), after which we give a more detailed account of the non(**Smz**) in Polish groups. We shall denote by **Smz** the ideal of strong measure zero subsets of $\mathbb{R}$. Concerning additivity of **Smz**, Carlson [9] in effect showed that $\mathsf{add}(\mathcal{N}) \leqslant \mathsf{add}(\mathbf{Smz})$, that $\mathsf{add}(\mathbf{Smz}) \leqslant \mathsf{non}(\mathcal{N})$ is a triviality, while Goldstern, Judah and Shelah [15] showed that consistently $\mathsf{cof}(\mathcal{M}) < \mathsf{add}(\mathbf{Smz})$, and, of course, Laver [28] that consistently $\mathsf{add}(\mathbf{Smz}) < \mathfrak{b}$ and Baumgartner [2] that consistently $\mathsf{add}(\mathbf{Smz}) < \mathsf{non}(\mathcal{N})$.

For cofinality of **Smz**, there are lower bounds $\mathsf{cov}(\mathcal{N})$ and $\mathsf{cov}(\mathcal{M})$ (see below), and it is folklore fact that assuming CH $\mathsf{cof}(\mathbf{Smz}) > \mathfrak{c}$, while the Borel conjecture produces models where $\mathsf{cof}(\mathbf{Smz}) = \mathfrak{c}$. Yorioka [51] and more recently Cardona [7] produced models of ZFC where $\mathsf{cof}(\mathbf{Smz}) < \mathfrak{c}$. According to our knowledge, it has not been subject to study if $\mathsf{add}(\mathbf{Smz}(\mathbb{G}))$, and/or $\mathsf{cof}(\mathbf{Smz}(\mathbb{G}))$ may vary depending on the Polish group in question.

It is, however, known that the uniformity numbers may differ depending on the group. There is also a surprising asymmetry between cov(**Smz**) and non(**Smz**). The trivial lower bound for cov(**Smz**) is $\mathsf{cov}(\mathcal{N})$ (every **Smz**-set has Lebesgue measure zero) and it also seems to be the best one. Pawlikowski [37] showed that $\mathsf{cov}(\mathbf{Smz}) < \mathsf{add}(\mathcal{M})$ is consistent. On the other hand, Cardona, Mejía and Riera-Marid [8] recently showed that $\mathsf{cov}(\mathbf{Smz}) = \omega_2 = \mathfrak{c}$ in the iterated Sacks model, hence, there seems to be very little in terms of upper bounds on cov(**Smz**), in particular, consistently $\mathsf{cof}(\mathcal{N}) < \mathsf{cov}(\mathbf{Smz})$.

Before moving on we shall mention some of the open problems concerning these invariants:

Question 4.1. (1) ([8]) Is it consistent that all four of the cardinal invariants corresponding to **Smz** have simultaneously different values?

(2) ([8]) Is it consistent that $\mathsf{add}(\mathbf{Smz}) < \min\{\mathsf{cov}(\mathbf{Smz}), \mathsf{non}(\mathbf{Smz})\}$?
(3) Do any of $\mathsf{add}(\mathbf{Smz}(\mathbb{G}))$, $\mathsf{cov}(\mathbf{Smz}(\mathbb{G}))$, $\mathsf{cof}\,\mathbf{Smz}(\mathbb{G}))$ depend on which Polish group one considers?

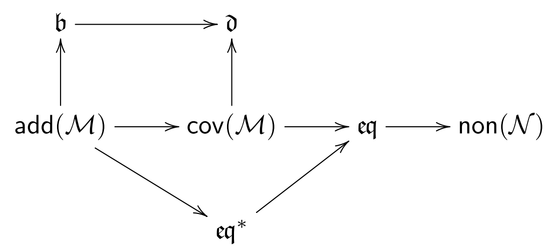
Finally, we arrive at the uniformity of $\mathbf{Smz}(\mathbb{G})$ which we shall discuss in considerably more detail. Two more invariants are required here (see [31, 29, 1, 19]):

$$\mathfrak{eq} = \min\{|F| : F \subseteq \omega^\omega \text{ bounded}, \forall g \in \omega^\omega\ \exists f \in F\ \forall n \in \omega\ f(n) \neq g(n)\}.$$
$$\mathfrak{eq}^* = \min\{|F| : F \subseteq \omega^\omega \text{ bounded},$$
$$\forall g, h \in \omega^\omega\ \exists f \in F\ \exists^\infty n\ \forall k \in [h(n), h(n+1))\ f(n) \neq g(n)\}.$$

It is a theorem of Bartoszyński [1, 2.4.1] that omitting "bounded" from the definition of $\mathfrak{eq}$ yields $\mathsf{cov}(\mathcal{M})$, i.e.

$$\mathsf{cov}(\mathcal{M}) = \min\{|F| : F \subseteq \omega^\omega\ \forall g \in \omega^\omega\ \exists f \in F\ \forall n \in \omega\ f(n) \neq g(n)\}.$$

The following diagram (see [1, 18] for proofs) summarizes the provable inequalities between the cardinal invariants mentioned (the arrows point from the smaller to the larger cardinal).



In addition, $\mathsf{add}(\mathcal{M}) = \min\{\mathfrak{b}, \mathfrak{eq}\} = \min\{\mathfrak{b}, \mathfrak{eq}^*\}$, while $\mathsf{cov}(\mathcal{M}) < \min\{\mathfrak{d}, \mathfrak{eq}\}$ is consistent with ZFC by a theorem of Goldstern, Judah and Shelah [15]. For any separable metric space X there are upper and lower bounds for $\mathsf{non}(\mathbf{Smz}(X))$ given by Rothberger [40] and Szpilrajn [46], respectively. The uniformity invariant $\mathsf{non}(\mathbf{Smz}(X))$ for $X = 2^\omega$ and $X = \omega^\omega$ was calculated by Bartoszyński [1], and Fremlin and Miller [30], respectively.

Theorem 4.2. *Let X be a separable metric space that is not* $\mathbf{Smz}$*.*
(i) ([40]) $\mathsf{cov}(\mathcal{M}) \leqslant \mathsf{non}(\mathbf{Smz}(X))$,
(ii) ([46]) *if X is not of universal measure zero*[2]*, then* $\mathsf{non}(\mathbf{Smz}(X)) \leqslant \mathsf{non}(\mathcal{N})$,
(iii) ([1]) $\mathsf{non}(\mathbf{Smz}(\omega^\omega)) = \mathsf{cov}(\mathcal{M})$ *and*
(iv) ([30]) $\mathsf{non}(\mathbf{Smz}(2^\omega)) = \mathfrak{eq}$.

Proof. (i) by now is standard: Given a separable metric space X and a sequence $\{\varepsilon_n : n \in \omega\}$, pick a countable dense set $\{d_n : n \in \omega\} \subseteq X$. For every one to one function $f \in \omega^\omega$ let $U_f = \bigcup_{n\in\omega} B(x_n, \varepsilon_{f_n})$. Now, assume that $|X| < \mathsf{cov}(\mathcal{M})$. To finish the proof it suffices to note that for every $x \in X$ the set $N_x = \{f \in \omega^\omega : f$ is one-to-one and $x \notin U_f\}$ is nowhere dense in the (closed) subspace of ω^ω consisting of one-to-one functions.

To see (ii) recall that every $\mathbf{Smz}$ set is of universal measure zero by Proposition 2.6, so it suffices to show that $\mathsf{non}(\mathcal{N})$ is the minimal size of a space which is

[2]Recall that a metric space X is of *universal measure zero* if there is no probability Borel measure on X vanishing on singletons.

not of universal measure zero. To see this note that any diffuse Borel probability measure μ on a separable metric space X extends to a finite Borel diffuse probability measure $\overline{\mu}$ on its completion $\hat{X}$ by putting $\overline{\mu}(A) = \mu(A \cap X)$. Now, by a theorem of Parthasarathy [35] there is a measure preserving Borel isomorphism between $\hat{X}$ with $\overline{\mu}$ and $[0,1]$ equipped with the Lebesgue measure, hence $|X| \geqslant \mathsf{non}(\mathcal{N})$.

For (iii) it suffices to see that $\mathsf{non}(\mathbf{Smz}(\omega^\omega)) \leqslant \mathsf{cov}(\mathcal{M})$. Let $F \subseteq \omega^\omega$ be such that $|F| = \mathsf{cov}(\mathcal{M})$ and $\forall g \in \omega^\omega\ \exists f \in F\ \forall n \in \omega\ f(n) \neq g(n)$. We shall show that $F \notin \mathbf{Smz}(\omega^\omega)$. Let $\langle s_n : n \in \omega\rangle$ be a sequence of elements of $2^{<\omega}$ such that $|s_n| = n+1$. Define $g \in \omega^\omega$ by putting $g(n) = s_n(n)$. Then there is an $f \in F$ such that $f(n) \neq g(n)$ for all $n \in \omega$. This, however, means, that $s_n \not\subseteq f$ for any $n \in \omega$. That is no sequence of open sets of diameter $\frac{1}{2^{n+1}}$ covers F.

The proof of (iv) is similar. First we shall show that $\mathsf{non}(\mathbf{Smz}(2^\omega)) \leqslant \mathfrak{eq}$. To that end let $X \subseteq 2^\omega$ be of size less that $\mathfrak{eq}$, and let a sequence $\langle \varepsilon_n : n \in \omega\rangle$ of positive real numbers be given. Let $h \in \omega^\omega$ be such that $\frac{1}{2^{h(n)}} \leqslant \varepsilon_n$ for every $n \in \omega$. For each $n \in \omega$ enumerate 2^n – the set of binary sequences of length n - as $\{s^n_m : m < 2^\omega\}$. For every $x \in X$ let $f_x \in \omega^\omega$ be defined by

$$f_x(n) = m \text{ if and only if } x{\restriction}h(n) = s^{h(n)}_m.$$

Then $f_x(n) \leqslant 2^{h(n)}$ for every $x \in X$ and $n \in \omega$. As $|X| < \mathfrak{eq}$, there is a $g \in \omega^\omega$ such that $f_x \cap g \neq \emptyset$ for every $x \in X$, and without loss of generality, $g(n) \leqslant 2^{h(n)}$ for every $n \in \omega$ (values above are irrelevant, and can be changed). Then $\langle\langle s^{h(n)}_{g(n)}\rangle : n \in \omega\rangle$ covers X.

On the other hand, assume that $F \subseteq \omega^\omega$ is a bounded family of size less than $\mathsf{non}(\mathbf{Smz}(2^\omega))$. Let $h \in \omega^\omega$ be such that $f(n) \leqslant 2^{h(n)}$ for every $f \in F$ and $n \in \omega$. Let $\{s^n_m : m < 2^\omega\}$ enumerate 2^n as above. For every $f \in \omega^\omega$ let

$$x_f = s^{h(0)}_{f(0)}{}^\frown s^{h(1)}_{f(1)}{}^\frown s^{h(2)}_{f(2)}{}^\frown \dots$$

and consider the set $X = \{x_f : f \in F\}$. As X is $\mathbf{Smz}$, there is a sequence $\langle t_n : n \in \omega\rangle$ such that

(1) $t_n \in 2^{\sum_{i \leqslant n} h(n)}$, and
(2) $\forall f \in F\ \exists n \in \omega\ t_n \subseteq x_f$.

Let $g \in \omega^\omega$ be such that $g(n) \leqslant 2^{h(n)}$ for all $n \in \omega$, and $g(n) = m$ whenever $t_{n+1} = t_n{}^\frown s^{h(n)}_m$. Note that $g(n) = f(n)$ whenever $t_n \subseteq x_f$. □

In particular, the theorem evaluates $\mathsf{non}(\mathbf{Smz})$ for two groups: the compact boolean group 2^ω and the Baer-Specker group $\mathbb{Z}^\omega$. In order to extend these to a wider class of groups we shall need two easy observations:

As mentioned before, strong measure zero is a uniform property, in particular, a uniformly continuous image of a **Smz** set is **Smz** and, on the other hand, if X uniformly embeds into Y, then any set $A \subseteq X$ that is not **Smz** in X is not **Smz** in Y either. It follows that:

Lemma 4.3. (i) *If a space Y is a uniformly continuous image of a space X then* $\mathsf{non}(\mathbf{Smz}(X)) \leqslant \mathsf{non}(\mathbf{Smz}(Y))$.
(ii) *If X uniformly embeds into Y, then* $\mathsf{non}(\mathbf{Smz}(X)) \geqslant \mathsf{non}(\mathbf{Smz}(Y))$.

Lemma 4.4. *A* CLI *Polish group is either locally compact, or contains a uniform copy of ω^ω.*

Proof. Let $\mathbb{G}$ be a group equipped with a complete, left-invariant metric d. Assuming $\mathbb{G}$ is not locally compact no open set is totally bounded, hence for every $\varepsilon > 0$ exists $\delta > 0$ such that the ball $B(1, \varepsilon)$ contains an infinite set of points that are pairwise at least δ apart. Using this fact construct, for each $n \in \omega$, $\varepsilon_n > 0$ and an infinite set $\{x_n^i : i \in \omega\} \subseteq B(1, \varepsilon_n)$ such that if $i \neq j$ then $d(x_n^i, x_n^j) > 5\varepsilon_{n+1}$. For $s \in \omega^n$ let $y_s = x_0^{s(0)} \cdot x_1^{s(1)} \cdot x_{n-2}^{s(n-2)} \cdot \cdots \cdot x_{n-1}^{s(n-1)}$. The construction ensures that for any $f \in \omega^\omega$ the sequence $\langle y_{f \restriction n} : n \in \omega \rangle$ is Cauchy. Let z_f be its limit. It is easy to check that since d is left-invariant, the mapping $f \mapsto z_f$ is a uniform embedding of ω^ω into $\mathbb{G}$. □

Theorem 4.5. *Let $\mathbb{G}$ be a* CLI *Polish group.*

(i) *If $\mathbb{G}$ is locally compact, then* $\mathsf{non}(\mathbf{Smz}(\mathbb{G})) = \mathfrak{eq}$*,*
(ii) *if $\mathbb{G}$ is not locally compact, then* $\mathsf{non}(\mathbf{Smz}(\mathbb{G})) = \mathsf{cov}(\mathcal{M})$*.*

Proof. As every Polish group contains a (uniform) copy of 2^ω,

$$\mathsf{cov}(\mathcal{M}) \leqslant \mathsf{non}(\mathbf{Smz}(\mathbb{G})) \leqslant \mathfrak{eq}$$

for every Polish group $\mathbb{G}$. Similarly, $\mathsf{non}(\mathbf{Smz}(\mathbb{G})) \leqslant \mathsf{cov}(\mathcal{M})$ whenever $\mathbb{G}$ contains a uniform copy of ω^ω by Theorem 4.2. In particular, $\mathsf{non}(\mathbf{Smz}(\mathbb{G})) = \mathsf{cov}(\mathcal{M})$ if $\mathbb{G}$ is a non-locally compact CLI group.

So all that remains to be seen is that $\mathfrak{eq} \leqslant \mathsf{non}(\mathbf{Smz}(\mathbb{G}))$ for any locally compact Polish group $\mathbb{G}$. Write $\mathbb{G}$ as an increasing union of compact subsets K_n, $n \in \omega$. As each K_n is a uniformly continuous image of 2^ω (see e.g. [22, Theorem 4.18]), every subset of K_n which is not of strong measure zero has size at least $\mathfrak{eq}$ by Lemma 4.3(i) and Theorem 4.2(iii). On the other hand, as $\mathbf{Smz}(\mathbb{G})$ is a σ-ideal (Proposition 2.1), every subset of $\mathbb{G}$ which is not $\mathbf{Smz}$ has a non-$\mathbf{Smz}$ intersection with one of the k_n's, hence $\mathfrak{eq} \leqslant \mathsf{non}(\mathbf{Smz}(\mathbb{G}))$. □

It is, of course, a natural question whether the result of the Theorem (or of the preceding lemma) remains true also for Polish groups which are not CLI.

The final remark of this section deals with *transitive covering for category* (considered by Bartoszyński [1, 2.7] for 2^ω, and Miller and Steprāns [31] for general Polish group and further studied in [19]):

$$\mathsf{cov}^*(\mathcal{M}(\mathbb{G})) = \min\{|A| : \ A \subseteq \mathbb{G} \text{ and } A \cdot M = \mathbb{G} \text{ for some meager set } M \subseteq \mathbb{G}\}$$

By Theorem 3.5, for a locally compact group $\mathbb{G}$ strong measure zero sets coincide with the sets whose meager translates do not cover $\mathbb{G}$, hence, in particular, $\mathsf{non}(\mathbf{Smz}(\mathbb{G})) = \mathsf{cov}^*(\mathcal{M}(\mathbb{G}))$ for every locally compact group $\mathbb{G}$. It follows from Prikry's Proposition 3.1 that $\mathsf{cov}^*(\mathcal{M}(\mathbb{G})) \leqslant \mathsf{non}(\mathbf{Smz}(\mathbb{G}))$, for every Polish group $\mathbb{G}$, hence $\mathsf{cov}(\mathcal{M}) \leqslant \mathsf{cov}^*(\mathcal{M}(\mathbb{G})) \leqslant \mathfrak{eq}$ for any Polish group. As $\mathsf{non}(\mathbf{Smz}(\mathbb{G})) = \mathsf{cov}(\mathcal{M})$ for all CLI groups which are not locally compact, we can conclude that:

Corollary 4.6. $\mathsf{non}(\mathbf{Smz}(\mathbb{G})) = \mathsf{cov}^*(\mathcal{M}(\mathbb{G}))$ *for any* CLI *group.*

The conjecture is that the two numbers coincide for any Polish group.

Question 4.7. Is $\mathsf{cov}^*(\mathcal{M}(\mathbb{G})) = \mathsf{cov}(\mathcal{M})$ for all Polish groups which are not locally compact?

5. Strong measure zero and Hausdorff measures

As mentioned above, there is a profound connection between strong measure zero and Hausdorff measures. The following characterizations of strong measure zero in terms of Hausdorff measures and dimensions were proved in [54]. They are based on a classical Besicovitch result [4, 5].

Hausdorff measure. Before getting any further we need to review Hausdorff measure and dimension. We set up the necessary definitions and recall relevant facts.

A non-decreasing, right-continuous function $h : [0, \infty) \to [0, \infty)$ such that $h(0) = 0$ and $h(r) > 0$ if $r > 0$ is called a *gauge*. Gauges are often ordered as follows, cf. [39]:

$$g \prec h \quad \overset{\text{def}}{\equiv} \quad \lim_{r \to 0+} \frac{h(r)}{g(r)} = 0.$$

Notice that for any sequence $\langle h_n : n \in \omega \rangle$ of gauges there is a gauge h such that $h \prec h_n$ for all n.

Given $\delta > 0$, call a cover $\mathcal{A}$ of a set $E \subseteq X$ a *δ-fine cover* if $\forall A \in \mathcal{A}$ $\operatorname{diam} A \leqslant \delta$. If h is a gauge, the *h-dimensional Hausdorff measure* $\mathcal{H}^h(E)$ of a set $E \subseteq X$ is defined thus: For each $\delta > 0$ define

$$\mathcal{H}^h_\delta(E) = \inf \left\{ \sum_{n \in \omega} h(\operatorname{diam} E_n) : \{E_n\} \text{ is a countable } \delta\text{-fine cover of } E \right\}$$

and let

$$\mathcal{H}^h(E) = \sup_{\delta > 0} \mathcal{H}^h_\delta(E).$$

In the common case of $h(r) = r^s$ for some $s > 0$, we write $\mathcal{H}^s$ for $\mathcal{H}^h$, and likewise for $\mathcal{H}^h_\delta$.

Properties of Hausdorff measures are well-known. The following, including the two lemmas, can be found e.g. in [39]. The restriction of $\mathcal{H}^h$ to Borel sets is a G_δ-regular Borel measure. Recall that a sequence of sets $\langle E_n : n \in \omega \rangle$ is termed a *λ-cover* of $E \subseteq X$ if every point of E is contained in infinitely many E_n's.

Theorem 5.1 ([4, 5])**.** *A metric space X is* **Smz** *if and only if $\mathcal{H}^h(X) = 0$ for each gauge h.*

Proof. Suppose first that X is **Smz**. Let h be a gauge. For each $\delta > 0$ pick a sequence $\langle \varepsilon_n \rangle$ such that $0 < \varepsilon_n < \delta$ and $h(\varepsilon_n) < \delta 2^{-n}$. There is a cover $\{U_n\}$ of X such that $\operatorname{diam} U_n < \varepsilon_n$ for all n. Obviously $\sum h(\operatorname{diam} U_n) < \delta$ and it follows that $\mathcal{H}^h_\delta(X) \leqslant \delta$. Let $\delta \to 0$ to get $\mathcal{H}^h(X) = 0$.

Now suppose that $\mathcal{H}^h(X) = 0$ for every gauge. Let $\langle \varepsilon_n \rangle$ be a sequence of positive numbers. Choose a gauge h such that $h(\varepsilon_n) > \frac{1}{n}$. Since $\mathcal{H}^h(X) = 0$, there is a countable cover $\{U_n\}$ such that $\sum h(\operatorname{diam} U_n) < 1$. As h is right-continuous, there are $\delta_n > \operatorname{diam} U_n$ such that $\sum h(\delta_n) < 1$. Since $\delta_n > 0$, rearranging the sequence we may suppose that δ_n decrease. Therefore $nh(\delta_n) \leqslant \sum_{i<n} h(\delta_n) < 1$. It follows that $h(\delta_n) < \frac{1}{n} < h(\varepsilon_n)$ and consequently $\delta_n < \varepsilon_n$ and in particular $\operatorname{diam} U_n < \varepsilon_n$, as required. □

We will need a cartesian product inequality. Given two metric spaces X and Y with respective metrics d_X and d_Y, provide the cartesian product $X \times Y$ with the maximum metric

$$d\big((x_1, y_1), (x_2, y_2)\big) = \max(d_X(x_1, x_2), d_Y(y_1, y_2)). \tag{5}$$

A gauge h satisfies the *doubling condition* or h is *doubling* if $\overline{\lim}_{r\to 0}\frac{h(2r)}{h(r)}<\infty$.

Lemma 5.2 ([23, 17])**.** *Let X,Y be metric spaces, g a gauge and h a doubling gauge. Then $\mathcal{H}^h(X)\,\mathcal{H}^g(Y)\leqslant\mathcal{H}^{hg}(X\times Y)$.*

The following lemma on uniformly continuous mappings is well-known, see, e.g., [39, Theorem 29].

Lemma 5.3. *Let $f:(X,d_X)\to(Y,d_Y)$ be a uniformly continuous mapping and g its modulus, i.e., $d_Y(f(x),f(y))\leqslant g(d_X(x,y))$ for all $x,y\in X$. Then $\mathcal{H}^h(f(X))\leqslant\mathcal{H}^{h\circ g}(X)$ for any gauge h.*

Recall that the *Hausdorff dimension* of X is defined by

$$\dim_{\mathsf{H}} X=\sup\{s>0:\mathcal{H}^s(X)=\infty\}=\inf\{s>0:\mathcal{H}^s(X)=0\}.$$

Properties of Hausdorff dimension are well-known. In particular, $\dim_{\mathsf{H}} X=0$ if X is countable; and if $f:X\to Y$ is Lipschitz, then $\dim_{\mathsf{H}} f(X)\leqslant\dim_{\mathsf{H}} X$.

Corollary 5.4 ([54])**.** *Let X be a metric space. The following are equivalent.*

(i) *X is* **Smz***,*
(ii) *$\dim_{\mathsf{H}} f(X)=0$ for each uniformly continuous mapping f on X,*
(iii) *$\dim_{\mathsf{H}}(X,\rho)=0$ for each uniformly equivalent metric ρ on X.*

Proof. (i)⇒(ii) Let $s>0$ be arbitrary. Let $f:X\to Y$ be uniformly continuous and let g be the modulus of f. Define $h(x)=(g(x))^s$. Lemma 5.3(i) yields $\mathcal{H}^s(f(X))\leqslant\mathcal{H}^h(X)$. By the above theorem $\mathcal{H}^h(X)=0$ and thus $\mathcal{H}^s(f(X))=0$. Since this holds for all $s>0$, it follows that $\dim_{\mathsf{H}} f(X)=0$.

(ii)⇒(iii) is trivial.

(iii)⇒(i) Denote by d the metric of X. Let h be a gauge. Choose a strictly increasing, convex (and in particular subadditive) gauge g such that $g\prec h$. The properties of g ensure that $\rho(x,y)=g(d(x,y))$ is a uniformly equivalent metric on X. The identity map $\mathrm{id}_X:(X,\rho)\to(X,d)$ is of course uniformly continuous and its modulus is g^{-1}, the inverse of g. Hence by Lemma 5.3(i) $\mathcal{H}^h(X,d)\leqslant\mathcal{H}^{h\circ g^{-1}}(X,\rho)$. Since $g\prec h$, we have $\mathcal{H}^{h\circ g^{-1}}(X,\rho)\leqslant\mathcal{H}^1(X,\rho)$ and $\mathcal{H}^1(X,\rho)=0$ by (ii). Thus $\mathcal{H}^h(X,d)=0$. □

Our next goal is to characterize **Smz** by behavior of cartesian products. Recall that for $p\in 2^{<\omega}$, $\langle p\rangle=\{x\in 2^\omega:p\subseteq x\}$ denotes the cone determined by p and for $T\subseteq 2^{<\omega}$ we let $\langle T\rangle=\bigcup_{p\in T}\langle p\rangle$.

The coordinatewise addition modulo 2 makes 2^ω a compact topological group. Routine proofs show that in the metric of the least difference (defined in the introduction) the 1-dimensional Hausdorff measure $\mathcal{H}^1$ coincides on Borel sets with its Haar measure, i.e., the usual product measure on 2^ω. In particular $\mathcal{H}^1(2^\omega)=1$.

We consider the important σ-ideal $\mathcal{E}$ on 2^ω generated by closed null sets, i.e., the ideal of all subsets of 2^ω that are contained in an F_σ set of Haar measure zero.

Lemma 5.5. (i) *For each $I\in[\omega]^\omega$, the set $C_I=\{x\in 2^\omega:x{\restriction}I\equiv 0\}$ is in $\mathcal{E}$.*
(ii) *For each $h\prec 1$ there is $I\in[\omega]^\omega$ such that $\mathcal{H}^h(C_I)>0$.*

Proof. (i) Let $I\in[\omega]^\omega$. For each $n\in\omega$, the family $\{\langle p\rangle:p\in C_I{\restriction}n\}$ is obviously a 2^{-n}-fine cover of C_I of cardinality $2^{|n\setminus I|}$. Therefore $\mathcal{H}^1_{2^{-n}}(C_I)\leqslant 2^{|n\setminus I|}2^{-n}=2^{-|n\cap I|}$. Hence $\mathcal{H}^1(C_I)\leqslant\lim_{n\to\infty}2^{-|n\cap I|}=0$.

(ii) $h \prec 1$ yields $\frac{h(2^{-n})}{2^{-n}} \to \infty$. Therefore there is $I \in [\omega]^\omega$ sparse enough to satisfy $2^{|n\cap I|} \leqslant \frac{h(2^{-n})}{2^{-n}}$, i.e., $2^{-|n\setminus I|} \leqslant h(2^{-n})$ for all $n \in \omega$. Consider the product measure λ on C_I given as follows: If $p \in 2^n$ and $\langle p\rangle \cap C_I \neq \emptyset$, put $\lambda(\langle p\rangle \cap C_I) = 2^{-|n\setminus I|}$. Straightforward calculation shows that $h(\operatorname{diam} E) \geqslant \lambda(E)$ for each $E \subseteq C_I$. Hence $\sum_n h(\operatorname{diam} E_n) \geqslant \sum_n \lambda(E_n) \geqslant \lambda(C_I) = 1$ for each cover $\{E_n\}$ of C_I and $\mathcal{H}^h(C_I) \geqslant 1$ follows. □

Theorem 5.6. *The following are equivalent.*

(i) *X is* **Smz**,
(ii) *$\mathcal{H}^h(X\times Y) = 0$ for every gauge h and every compact metric space Y such that $\mathcal{H}^h(Y) = 0$,*
(iii) *$\mathcal{H}^1(X \times E) = 0$ for every $E \in \mathcal{E}$.*

Proof. (i)⇒(ii): Suppose X is **Smz** and let Y be compact. Let $\delta > 0$. Since $\mathcal{H}^h(Y) = 0$, for each $j \in \omega$ there is a finite cover $\mathcal{U}_j$ of Y such that $\sum_{U\in\mathcal{U}_j} h(\operatorname{diam} U) < 2^{-j-1}\delta$. We may also assume that $\operatorname{diam} U < \delta$ for all $U \in \mathcal{U}_j$.

Let $\varepsilon_j = \min\{\operatorname{diam} U : U \in \mathcal{U}_j\}$. Since X is **Smz**, there is a cover $\{V_j\}$ of X such that $\operatorname{diam} V_j \leqslant \varepsilon_j$. Define

$$\mathcal{W} = \{V_j \times U : j \in \omega,\, U \in \mathcal{U}_j\}.$$

$\mathcal{W}$ is obviously a cover of $X \times Y$. The choice of ε_j yields $\operatorname{diam}(V_j \times U) = \operatorname{diam} U$ for all j and $U \in \mathcal{U}_j$. Therefore

$$\sum_{W\in\mathcal{W}} h(\operatorname{diam} W) = \sum_{j\in\omega}\sum_{U\in\mathcal{U}_j} h(\operatorname{diam} U) < \sum_{j\in\omega} 2^{-j-1}\delta = \delta.$$

It follows that $\mathcal{H}^h_\delta(X \times Y) < \delta$, and $\mathcal{H}^h(X \times Y) = 0$ obtains by letting $\delta \to 0$.

(ii)⇒(iii)⇒(iv) is trivial.

(iv)⇒(i): Suppose X is not **Smz**. We will show that $\mathcal{H}^1(X \times E) > 0$ for some $E \in \mathcal{E}$. By assumption and Theorem 5.4 there is a gauge h such that $\mathcal{H}^h(X) > 0$. We may assume h be concave and $h(r) \geqslant \sqrt{r}$. In particular, by concavity of h the function $g(r) = r/h(r)$ is increasing. Moreover, $h(r) \geqslant \sqrt{r}$ yields $\lim_{r\to 0} g(r) = 0$, i.e., g is a gauge, and $g \prec 1$. Further, $g(2r) = 2r/h(2r) \leqslant 2r/h(r) = 2g(r)$, i.e., g is doubling.

Use Lemma 5.5(ii) to find $I \in [\omega]^\omega$ such that $\mathcal{H}^g(C_I) > 0$ and let $E = C_I$. By Lemma 5.5(i), $E \in \mathcal{E}$. Since g is doubling, Lemma 5.2 applies:

$$\mathcal{H}^1(X \times C_I) = \mathcal{H}^{h\cdot g}(X \times C_I) \geqslant \mathcal{H}^h(X) \cdot \mathcal{H}^g(C_I) > 0. \qquad \square$$

Corollary 5.7. *If X is* **Smz**, *then $\dim_{\mathsf{H}} X\times Y = \dim_{\mathsf{H}} Y$ for every compact metric space Y.*

Note that this consistently fails when we drop the assumption that Y is compact: By a classical example (cf. [12, 534P]), if $\mathsf{cov}(\mathcal{M}) = \mathfrak{c}$, then there is a **Smz** set $X \subseteq \mathbb{R}$ such that $X + X = \mathbb{R}$. Since $X + X$ is a Lipschitz image of $X \times X$, we have $\dim_{\mathsf{H}} X \times X \geqslant \dim_{\mathsf{H}} X + X = 1$, while X is **Smz** and $\dim_{\mathsf{H}} X = 0$.

6. Sharp measure zero

Recall that a set S in a Polish group is called *meager-additive* if $S \cdot M$ is meager for every meager set M. This is obviously a strengthening of the "algebraic" characterization of **Smz** in the Galvin-Mycielski-Solovay Theorem. Meager-additive sets, in particular in 2^ω, have received a lot of attention, e.g., in [42, 1, 33, 47].

Very recently it was shown that meager-additive sets are characterized by a combinatorial condition very similar to the definition of **Smz** and also in terms of Hausdorff measures. In this section we will have a look at these descriptions.

Definition 6.1. A set $S \subseteq X$ in a complete metric space X has *sharp measure zero* if for every gauge h there is a σ-compact set $K \supseteq S$ such that $\mathcal{H}^h(K) = 0$.

We first work towards an intrinsic definition equivalent to the one above. The following variation of Hausdorff measure seems to be the right notion for that. Let h be a gauge. For each $\delta > 0$ define

$$\overline{\mathcal{H}}^h_\delta(E) = \inf\left\{\sum_{n=0}^{N} h(\operatorname{diam} E_n) : \{E_n : n \leqslant N\} \text{ is a \textit{finite} } \delta\text{-fine cover of } E\right\}$$

and let

$$\overline{\mathcal{H}}^h_0(E) = \sup_{\delta>0} \overline{\mathcal{H}}^h_\delta(E).$$

Note the striking similarity with the Hausdorff measure. The only difference is that only finite covers are taken into account. It is easy to check that $\overline{\mathcal{H}}^h_0$ is finitely subadditive. However, it is not a measure, since it may (due to the finite covers) lack σ-additivity. To turn it into a measure we need to apply the operation known as Munroe's *Method I construction* (cf. [32] or [39]):

$$\overline{\mathcal{H}}^h(E) = \inf\Big\{\sum_{n\in\omega} \overline{\mathcal{H}}^h_0(E_n) : E \subseteq \bigcup_{n\in\omega} E_n\Big\}.$$

Thus the defined set function $\overline{\mathcal{H}}^h$ is indeed an outer measure whose restriction to Borel sets is a Borel measure. We will called it *h-dimensional upper Hausdorff measure.*

Upper Hausdorff measures behave much like Hausdorff measures. We list some important properties of upper Hausdorff measures. We refer to [54] for details.

Denote $\mathcal{N}_\sigma(\overline{\mathcal{H}}^h_0)$ the smallest σ-additive ideal that contains all sets E with $\overline{\mathcal{H}}^h_0(E) = 0$. Note that while $E \in \mathcal{N}_\sigma(\overline{\mathcal{H}}^h_0) \Rightarrow \overline{\mathcal{H}}^h(E) = 0$, the reverse implication in general fails. Write $E_n \nearrow E$ to denote that $\langle E_n : n \in \omega\rangle$ is an increasing sequence of sets with union E. The following lists some basic features of $\overline{\mathcal{H}}^h$ and $\overline{\mathcal{H}}^h_0$.

Lemma 6.2 ([54])**.** *Let h be a gauge and E a set in a metric space.*

(i) *If $\overline{\mathcal{H}}^h_0(E) < \infty$, then E is totally bounded.*
(ii) *$\overline{\mathcal{H}}^h_0(E) = \overline{\mathcal{H}}^h_0(\overline{E})$.*
(iii) *$\overline{\mathcal{H}}^h_0(E) = \mathcal{H}^h(E)$ if E is compact.*
(iv) *If $E \in \mathcal{N}_\sigma(\overline{\mathcal{H}}^h_0)$, then $\overline{\mathcal{H}}^h(E) = 0$.*
(v) *If X is complete, $E \subseteq X$ and $E \in \mathcal{N}_\sigma(\overline{\mathcal{H}}^h_0)$, then there is a σ-compact set $K \supseteq E$ such that $\mathcal{H}^h(K) = 0$.*
(vi) *If X is complete and $E \subseteq X$, then $\overline{\mathcal{H}}^h(E) = \inf\{\mathcal{H}^h(K) : K \supseteq E$ is σ-compact$\}$.*
(vii) *In particular $\overline{\mathcal{H}}^h(E) = \mathcal{H}^h(E)$ if E is σ-compact.*
(viii) *If $g \prec h$ and $\overline{\mathcal{H}}^g(E) < \infty$, then $E \in \mathcal{N}_\sigma(\overline{\mathcal{H}}^h_0)$; in particular, $\overline{\mathcal{H}}^h(E) = 0$.*

Lemma 6.3 ([54])**.** *$E \in \mathcal{N}_\sigma(\overline{\mathcal{H}}^h_0)$ if and only if E has a γ-groupable cover $\langle U_n : n \in \omega\rangle$ such that $\sum_{n\in\omega} h(\operatorname{diam} U_n) < \infty$.*

Proof. Suppose that $E \in \mathcal{N}_\sigma(\overline{\mathcal{H}}^h_0)$. Let $E_n \nearrow E$ be such that $\overline{\mathcal{H}}^h_0(E_n) = 0$. For each n let $\mathcal{G}_n$ be a finite cover of E_n such that $\sum_{G \in \mathcal{G}_n} h(\operatorname{diam} G) < 2^{-n}$. Since the family $\mathcal{G} = \bigcup_n \mathcal{G}_n$ is obviously a γ-groupable cover, we are done.

In the opposite direction, suppose that $\langle U_n : n \in \omega\rangle$ is a γ-groupable cover $\langle U_n : n \in \omega\rangle$ such that $\sum_{n\in\omega} h(\operatorname{diam} U_n) < \infty$ with witnessing families $\mathcal{G}_j$. Let $E_k = \bigcap_{j \geqslant k} \bigcup \mathcal{G}_j$. Then $E = \bigcup_{k\in\omega} E_k$. For each k, the set E_k is covered by each $\mathcal{G}_j$, $j \geqslant k$, and $\sum_{G\in\mathcal{G}_j} h(\operatorname{diam} G)$ is as small as needed for j large enough. Hence $\overline{\mathcal{H}}^h_0(E_k) = 0$ and consequently $E \in \mathcal{N}_\sigma(\overline{\mathcal{H}}^h_0)$. □

It is straightforward from Lemma 6.2 that the following intrinsic definition of sharp measure zero is consistent with the one above.

Definition 6.4. A metric space X has sharp measure zero if $\overline{\mathcal{H}}^h(X) = 0$ for every gauge h. Sharp measure zero is abbreviated as $\mathsf{Smz}^\sharp$.

It is no surprise that Theorem 5.4 has a counterpart for $\mathsf{Smz}^\sharp$, with basically the same proof. Upper Hausdorff dimension is defined as expected:

$$\overline{\dim}_{\mathsf{H}} X = \sup\{s > 0 : \overline{\mathcal{H}}^s(X) = \infty\} = \inf\{s > 0 : \overline{\mathcal{H}}^s(X) = 0\},$$

see [54, 53] for more on the Hausdorff dimension.

Theorem 6.5 ([54])**.** *Let X be a metric space. The following are equivalent.*

- (i) *X is $\mathsf{Smz}^\sharp$,*
- (ii) *$\overline{\dim}_{\mathsf{H}} f(X) = 0$ for each uniformly continuous mapping f on X,*
- (iii) *$\overline{\dim}_{\mathsf{H}}(X, \rho) = 0$ for each uniformly equivalent metric ρ on X.*

It is straightforward from the definition and Theorem 6.5 that $\mathsf{Smz}^\sharp$ is a σ-additive property and that it is preserved by uniformly continuous mappings.

Sharp measure zero can be described in terms of covers. The description is strikingly similar to the Borel's definition of strong measure zero.

A countable cover $\{U_j\}$ of X is a called a *γ-cover* if each $x \in X$ belongs to all but finitely many U_j.

The following notion was studied, e.g., in [24]. A sequence $\langle W_n\rangle$ of sets in X is called a *γ-groupable cover* if there is a partition $\omega = I_0 \cup I_1 \cup I_2 \cup \dots$ into consecutive finite intervals (i.e. I_{j+1} is on the right of I_j for all j) such that the sequence $\langle \bigcup_{n\in I_j} W_n : j \in \omega\rangle$ is a γ-cover. The partition $\langle I_j\rangle$ will be occasionally called a *witnessing partition* and the finite families $\{U_n : n \in I_j\}$ will be occasionally called *witnessing families.*

Theorem 6.6 ([54])**.** *A metric space X is $\mathsf{Smz}^\sharp$ if and only if it has the following property: for every sequence $\langle \varepsilon_n : n \in \omega\rangle$ of positive real numbers there is a γ-groupable cover $\{U_n : n \in \omega\}$ of X such that* $\operatorname{diam} U_n \leqslant \varepsilon_n$ *for all n.*

Proof. The pattern of the proof is the same as that of the proof of the Besicovitch's theorem 5.1, but there are details that make it much more involved.

The forward implication is easy. Let h be a gauge. Pick $\varepsilon_n > 0$ such that $\sum_n h(\varepsilon_n) < \infty$. By assumption, there is a γ-groupable cover $\langle G_n\rangle$ such that $\operatorname{diam} G_n \leqslant \varepsilon_n$. Therefore $\sum_n h(\operatorname{diam} G_n) \leqslant \sum_n h(\varepsilon_n) < \infty$. Now apply Lemma 6.3 to conclude that $X \in \mathcal{N}_\sigma(\overline{\mathcal{H}}^h_0)$ and in particular $\overline{\mathcal{H}}^h(X) = 0$.

The reverse implication: Let $\langle \varepsilon_n\rangle \in (0,\infty)^\omega$. Choose a gauge g such that $g(\varepsilon_n) > \frac{1}{n}$ for all $n \geqslant 1$ and then a gauge $h \prec g$. Since $\overline{\mathcal{H}}^h(X) = 0$, Lemma 6.2(viii) yields

$X \in \mathcal{N}_\sigma(\overline{\mathcal{H}}^g_0)$, which in turn yields, with the aid of Lemma 6.3, a γ-groupable cover $\langle G_n : n \in \omega\rangle$ such that $\sum_n g(\operatorname{diam} G_n) < \infty$. Let $\{I_j : j \in \omega\}$ be the witnessing partition and $\mathcal{G}_j = \{G_n : n \in I_j\}$ the witnessing families.

We want to permute the cover so that diameters decrease. Some of the diameters may be 0. Also, the permutation may break down the witnessing families. We thus have to exercise some care.

For each n choose $\delta_n > \operatorname{diam} G_n$ so that $\sum_n g(\delta_n) < \infty$. Next choose an increasing sequence $\langle j_k : n \in \omega\rangle$ satisfying for all $k \in \omega$

(a) $\sum\{g(\delta_n) : n \in I_{j_k}\} < 2^{-k-1}$,
(b) $\max\{\delta_n : n \in I_{j_{k+1}}\} < \min\{\delta_n : n \in I_{j_k}\}$.

Let $I = \bigcup_{k\in\omega} I_{j_k}$. Rearrange G_n's within each group $\mathcal{G}_{j_k}$ so that $\langle \delta_n : n \in I_{j_k}\rangle$ form a non-increasing sequence. Together with (b) this ensures that the sequence $\langle \delta_n : n \in I\rangle$ is non-increasing.

For each $n \in \omega$ let $\widehat{n} \in I$ be the unique index such that $n = |I \cap \widehat{n}|$ and define $H_n = G_{\widehat{n}}$. It follows, with the aid of (a) and the definition of g, that for all $n \in \omega$

$$\begin{aligned} g(\operatorname{diam} H_n) = g(\operatorname{diam} G_{\widehat{n}}) \leqslant g(\delta_{\widehat{n}}) &\leqslant \frac{1}{n}\sum\{g(\delta_m) : m \in I, m \leqslant \widehat{n}\} \\ &\leqslant \frac{1}{n}\sum\{g(\delta_m) : m \in I\} \leqslant \frac{1}{n} < g(\varepsilon_n) \end{aligned}$$

and thus $\operatorname{diam} H_n \leqslant \varepsilon_n$ for all n. Moreover, the families $\mathcal{G}_{j_k}$, $k \in \omega$, witness that $\langle H_n : n \in \omega\rangle$ is a γ-groupable cover. □

Theorem 6.7 ([54])**.** *Let X be a metric space. The following are equivalent.*

(i) *X is $\mathsf{Smz}^\sharp$,*
(ii) *$\overline{\mathcal{H}}^h(X \times Y) = 0$ for each gauge h and Y such that $\overline{\mathcal{H}}^h_0(Y) = 0$,*
(iii) *$\overline{\mathcal{H}}^1(X \times E) = 0$ for each $E \in \mathcal{E}$.*

Proof. (i)⇒(ii): Suppose X is $\mathsf{Smz}^\sharp$. Let h be a gauge and $\overline{\mathcal{H}}^h_0(Y) = 0$. By Lemma 6.3 there is a γ-groupable cover $\mathcal{U}$ of Y such that $\sum_{U\in\mathcal{U}} h(\operatorname{diam} U) < \infty$. For each $U \in \mathcal{U}$ there is $\delta_U > \operatorname{diam} U$ such that $\sum_{U\in\mathcal{U}} h(\delta_U) < \infty$. Denote by $\mathcal{U}_j$ the witnessing families and let $\varepsilon_j = \min\{\delta_U : U \in \mathcal{U}_j\}$. Since X is $\mathsf{Smz}^\sharp$, Theorem 6.6 yields a γ-groupable cover $\langle V_j : n \in \omega\rangle$ of X such that $\operatorname{diam} V_j \leqslant \varepsilon_j$. Denote by $\mathcal{V}_k$ the witnessing families. Define a family of sets in $X \times Y$

$$\mathcal{W} = \{V_j \times U : j \in \omega,\, U \in \mathcal{U}_j\}.$$

It is routine to check that $\mathcal{W}$ is a γ-groupable cover of $F \times Y$. Since $\operatorname{diam}(V_j \times U) \leqslant \delta_U$ for all j and $U \in \mathcal{U}_j$ by the choice of ε_j, we have

$$\sum_{W\in\mathcal{W}} h(\operatorname{diam} W) \leqslant \sum_{U\in\mathcal{U}} h(\delta_U) < \infty.$$

Thus it follows from Lemma 6.3 that $X \times Y \in \mathcal{N}_\sigma(\overline{\mathcal{H}}^h_0)$ and in particular $\overline{\mathcal{H}}^h(X \times Y) = 0$.

(ii)⇒(iii) is trivial. The proof of (iii)⇒(i) is very much like that of Theorem 5.6. Suppose X is not $\mathsf{Smz}^\sharp$. We need to find $E \in \mathcal{E}$ such that $\overline{\mathcal{H}}^1(X \times E) > 0$. By assumption there is a gauge h such that $\overline{\mathcal{H}}^h(X) > 0$. We may suppose h is concave, and find a doubling gauge $g \prec 1$ such that $g(r)h(r) = r$. Then use Lemma 5.5(ii) to find $I \in [\omega]^\omega$ such that $\mathcal{H}^g(C_I) > 0$. We now need a product inequality on upper Hausdorff measures analogous to Lemma 5.2 proved in [54, 3.5,7.4].

Lemma ([54]). *Let X, Y be metric spaces and g a gauge and h a doubling gauge. Then $\mathcal{H}^h(X)\,\overline{\mathcal{H}}^g(Y) \leqslant \overline{\mathcal{H}}^{hg}(X \times Y)$.*

Using this lemma, we get

$$\overline{\mathcal{H}}^1(X \times C_I) = \overline{\mathcal{H}}^{h\cdot g}(X \times C_I) \geqslant \overline{\mathcal{H}}^h(X) \cdot \mathcal{H}^g(C_I) > 0. \qquad \square$$

As we already mentioned, under $\mathsf{cov}(\mathcal{M}) = \mathfrak{c}$ there is an example ([12, 534P]) of a **Smz** set $X \subseteq \mathbb{R}$ such that $X \times X$ is not **Smz**. Scheepers [41] examines thoroughly conditions imposed on a **Smz** set X that would ensure that a product of X with another **Smz** set is **Smz**. A recent roofing result claims that if one of the factors is **Smz**$^\sharp$, then the product is **Smz**.

Theorem 6.8 ([54]). (i) *If X and Y are* **Smz**$^\sharp$*, then $X \times Y$ is* **Smz**$^\sharp$*.*
(ii) *If X is* **Smz** *and Y is* **Smz**$^\sharp$*, then $X \times Y$ is* **Smz***.*

Proof. Suppose Y is **Smz**$^\sharp$. By Lemma 6.2(viii), $Y \in \mathcal{N}_\sigma(\overline{\mathcal{H}}^h_0)$ for all gauges h.

(i) If X is **Smz**$^\sharp$, then Theorem 6.7(ii) yields $\overline{\mathcal{H}}^h(X \times Y) = 0$ for all gauges h.

(ii) Let h be a gauge. Since Y is **Smz**$^\sharp$, it is σ-totally bounded and therefore there is a σ-compact set $K \supseteq Y$ in the completion of Y such that $\mathcal{H}^h(K) = 0$. Since X is **Smz**, Theorem 5.6(ii) yields $\mathcal{H}^h(X \times Y) = 0$, which is by Theorem 5.4(ii) enough. $\square$

This theorem, together with the above example, provides an easy argument that shows that consistently not every **Smz** set is **Smz**$^\sharp$: The **Smz** set X such that $X \times X$ is not **Smz** cannot be **Smz**$^\sharp$.

We illustrate the power of the theorem by the following

Corollary 6.9. *Let $X \subseteq \mathbb{R}^2$. The following are equivalent.*

(i) *X is* **Smz**$^\sharp$*,*
(ii) *all orthogonal projections of X on lines are* **Smz**$^\sharp$*,*
(iii) *at least two orthogonal projections of X on lines are* **Smz**$^\sharp$*.*

Proof. Since orthogonal projections are uniformly continuous, (i)⇒(ii) from preservation of **Smz**$^\sharp$ by uniformly continuous mappings. (ii)⇒(iii) is trivial. (iii)⇒(i): Let L_1, L_2 be two nonparallel lines and π_1, π_2 the corresponding orthogonal projections. *Mutatis mutandis* we may suppose that L_1 is the x-axis and L_2 is the y-axis. Thus $X \subseteq \pi_1 X \times \pi_2 X$. Theorem 6.8(ii) thus concludes the proof. $\square$

Theorem 6.8(ii) also raises the question whether a space whose product with any **Smz** set of reals is **Smz** has to be **Smz**$^\sharp$. As shown in [54], the answer is consistently no: The forcing extension constructed by Corazza in [10] we have the following. A similar observation was noted without proof in [33] and also in [49].

Proposition 6.10. *In the Corazza model there is a set $X \subseteq 2^\omega$ that is not* **Smz**$^\sharp$ *and yet $X \times Y$ is* **Smz** *for each* **Smz** *set $Y \subseteq 2^\omega$.*

7. Meager additive sets and sharp measure zero

We now look at the meager-additive sets in Polish groups and establish their surprising and profound connection with **Smz**$^\sharp$ sets. The theory nicely parallels the Galvin-Mycielski-Solovay Theorem. Most of the material of this section comes from [52] and [54].

Whenever $\mathbb{G}$ is a Polish group, $\mathsf{Smz}^\sharp(\mathbb{G})$ denotes the family of sharp measure zero sets with respect to any left-invariant metric. The notion od $\mathsf{Smz}^\sharp$ is of course a uniform invariant – it is neither a topological, nor a metric property. Therefore it does not matter which left-invariant metric we choose. The same proof shows that $\mathsf{Smz}^\sharp$ sets, just like Smz sets, form a bi-invariant σ-ideal.

Proposition 7.1. $\mathsf{Smz}^\sharp(\mathbb{G})$ *is a bi-invariant* σ*–ideal.*

Recall that if $\mathbb{G}$ is a Polish group, we denote by $\mathcal{M}(\mathbb{G})$, or simply by $\mathcal{M}$ if there is no danger of confusion, the ideal of meager subsets of $\mathbb{G}$.

Definition 7.2. A set $S \subseteq \mathbb{G}$ is called *meager-additive* (or $\mathcal{M}$*-additive*) if SM is meager for every meager set $M \subseteq \mathbb{G}$. The family of all meager-additive sets is denoted by $\mathcal{M}^*(\mathbb{G})$.

It is straightforward from the definition that

Proposition 7.3. $\mathcal{M}^*(\mathbb{G})$ *is a bi-invariant* σ*–ideal.*

The hard implication of Galvin-Mycielski-Solovay Theorem claims that if S is Smz set in a locally compact group, then $S \cdot M \neq \mathbb{G}$. The analogous statement for $\mathsf{Smz}^\sharp$ and meager-additive sets is about as hard as that.

Theorem 7.4 ([52])**.** *Let* $\mathbb{G}$ *be a locally compact Polish group. Then every* $\mathsf{Smz}^\sharp$ *set* $S \subseteq \mathbb{G}$ *is* $\mathcal{M}$*-additive, i.e.,* $\mathsf{Smz}^\sharp(\mathbb{G}) \subseteq \mathcal{M}^*(\mathbb{G})$*.*

Proof. The proof utilizes Lemma 3.3. Suppose $S \subseteq \mathbb{G}$ is a $\mathsf{Smz}^\sharp$ set and let $M \subseteq \mathbb{G}$ be meager. Let K_n be compact sets in $\mathbb{G}$ with $K_n \nearrow \mathbb{G}$ and let P_n be compact nowhere dense sets with $P_n \nearrow M$. Let $\{U_k\}$ be a countable base of $\mathbb{G}$. For each k choose $x_0^k \in \mathbb{G}$ and $\varepsilon_0^k > 0$ such that $B(x_0^k, \varepsilon_0^k) \subseteq U_k$ is compact, and let $C_k = B(x_0^k, \varepsilon_0^k)$. Use Lemma 3.3 to recursively construct a sequence $\langle \varepsilon_n : n \in \omega \rangle$ of positive numbers such that

$$\begin{aligned} &\forall n\ \forall i \leqslant n\ \forall x \in C_i\ \forall y \in K_n\ \exists z \in C_i \\ &\qquad B(z, \varepsilon_n) \subseteq B(x, \varepsilon_{n-1}) \setminus ((B(y, \varepsilon_n) \cap K_n) \cdot P_n). \end{aligned} \tag{6}$$

Since S is $\mathsf{Smz}^\sharp$, there is an γ-groupable cover $\{E_n\}$ of S such that $\operatorname{diam} E_n < \varepsilon_n$ for all n. Hence for each n there is y such that $E_n \subseteq B(y, \varepsilon_n)$. Therefore we may use (6) to construct for each k a sequence $\langle x_n^k : n \in \omega \rangle$ such that

$$B(x_{n+1}^k, \varepsilon_{n+1}) \subseteq B(x_n^k, \varepsilon_n) \setminus ((E_{n+1} \cap K_{n+1}) \cdot P_{n+1}). \tag{7}$$

It is easy to check that since $\{E_n\}$ is a γ-groupable cover of S and $K_n \nearrow \mathbb{G}$, the family $\{E_n \cap K_n\}$ is also a γ-groupable cover of S. Thus we might have supposed that $E_n \subseteq K_n$, and also that all E_n's are closed. Therefore (7) simplifies to

$$B(x_{n+1}^k, \varepsilon_{n+1}) \subseteq B(x_n^k, \varepsilon_n) \setminus (E_{n+1} \cdot P_{n+1}). \tag{8}$$

In particular, $B(x_n^k, \varepsilon_n)$ is a decreasing sequence of compact balls for all k and thus there is a point $x^k \in U_k$ such that

$$x^k \notin \bigcup_{n \in \omega} (E_n \cdot P_n). \tag{9}$$

Now construct a set $\widehat{S}$ as follows: Let $\mathcal{G}_j$ be the groups of E_n's witnessing to the γ-groupability of $\{E_n\}$. Put $G_n = \bigcap_{n \in \mathcal{G}_j} E_n$ and let $F_n = \bigcap_{i<n} G_i$ and $\widehat{S} = \bigcup_{n \in \omega} F_n$. It is clear that since E_n's are closed, the set $\widehat{S}$ is F_σ, and clearly $S \subseteq \widehat{S}$. Moreover,

routine calculation shows that $\widehat{S}\times M \subseteq \bigcup_n E_n \times P_n$. Therefore (9) yields $x^k \notin \widehat{S}\cdot M$ for all k. So letting $D = \{x^k : k \in \omega\}$, the set D is disjoint with $\widehat{S}\cdot M$ and it is dense in X. Since $\widehat{S}$ and M are σ-compact, so is the set $\widehat{S}\times M$. It follows that $\widehat{S}\cdot M$, being a continuous image of $\widehat{S}\times M$, is also σ-compact. In summary, $\widehat{S}\cdot M)$ is an F_σ set disjoint with a dense set, and is thus meager. □

One would expect that the reverse implication that parallels the trivial Proposition 3.1 of Prikry would be also very easy. Surprisingly, it is not easy at all. Only very recently it was proved in [52] that it holds for Polish groups that admit a (both-sided) invariant metric.

Recall that Polish groups that admit an invariant metric are referred to as TSI groups. Compact or abelian Polish groups are TSI; any invariant metric on a TSI group is complete.

Theorem 7.5 ([52])**.** *Let $\mathbb{G}$ be a* TSI *Polish group. If $S \subseteq \mathbb{G}$ is an $\mathcal{M}$-additive set, then S is* **Smz**$^\sharp$ *(in any metric on $\mathbb{G}$).*

Theorem 7.6. *Let $\mathbb{G}$ be a locally compact* TSI *Polish group. Then* **Smz**$^\sharp(\mathbb{G}) = \mathcal{M}^*(\mathbb{G})$.

It takes several pages to prove Theorem 7.5, in contrast to the five lines of the proof of Proposition 3.1. We present a proof of the particular case of $\mathbb{G} = 2^\omega$ that takes advantage of the regular combinatorial structure of the Cantor set and a deep characterization by Shelah of $\mathcal{M}$-additive sets ([42] or [1, Theorem 2.7.17]) and is thus much shorter.

Lemma 7.7 ([42])**.** *$X \subseteq 2^\omega$ is $\mathcal{M}$-additive if and only if*

$$\forall f \in \omega^{\uparrow\omega}\ \exists g \in \omega^\omega\ \exists y \in 2^\omega\ \forall x \in X\ \forall^\infty n\ \exists k$$
$$g(n) \leqslant f(k) < f(k+1) \leqslant g(n+1)\ \&\ x{\restriction}[f(k), f(k+1)) = y{\restriction}[f(k), f(k+1)).$$

Theorem 7.8. **Smz**$^\sharp(2^\omega) = \mathcal{M}^*(2^\omega)$.

Proof. Let $S \subseteq 2^\omega$ be $\mathcal{M}$-additive. Let h be a gauge. We will show that $\overline{\mathcal{H}}^h(S) = 0$. Define recursively $f \in \omega^{\uparrow\omega}$ subject to

$$2^{f(k)} \cdot h\big(2^{-f(k+1)}\big) \leqslant 2^{-k}, \quad k \in \omega.$$

By Lemma 7.7 there is $g \in \omega^\omega$ and $y \in 2^\omega$ such that

$$\text{(10)}\quad \forall x \in X\ \forall^\infty n\ \exists k \quad g(n) \leqslant f(k) < g(n+1)\ \&\ x{\restriction}[f(k), f(k+1)) = y{\restriction}[f(k), f(k+1)).$$

Define

$$\mathcal{B}_k = \big\{\langle p^\frown y{\restriction}[f(k), f(k+1))\rangle : p \in 2^{f(k)}\big\}, \qquad k \in \omega,$$
$$\mathcal{G}_n = \bigcup\big\{\mathcal{B}_k : g(n) \leqslant f(k) < g(n+1)\big\}, \qquad n \in \omega,$$
$$\mathcal{B} = \bigcup_{k\in\omega} \mathcal{B}_k = \bigcup_{n\in\omega} \mathcal{G}_n.$$

With this notation (10) reads

$$\text{(11)}\qquad \forall x \in X\ \forall^\infty n\ \exists G \in \mathcal{G}_n\ x \in G.$$

Since each of the families $\mathcal{G}_n$ is finite, it follows that $\mathcal{G}_n$'s witness that $\mathcal{B}$ is a γ-groupable cover of X. Using Lemma 6.3 (and Lemma 6.2(iv)) it remains to show that the Hausdorff sum $\sum_{B\in\mathcal{B}} h(\operatorname{diam} B)$ is finite. Note that the cones forming the families $B \in \mathcal{B}_k$ are actually balls of radius $2^{-f(k+1)}$, i.e., $\operatorname{diam} B = 2^{-f(k+1)}$ for all k and all $B \in \mathcal{B}_k$. Each of the cones is determined by one $p \in 2^{f(k)}$, therefore $|\mathcal{B}_k| = 2^{f(k)}$. Overall we have

$$\sum_{B\in\mathcal{B}} h(\operatorname{diam} B) = \sum_{k\in\omega}\sum_{B\in\mathcal{B}_k} h(\operatorname{diam} B) = \sum_{k\in\omega} 2^{f(k)}\cdot h(2^{-f(k+1)}) \leqslant \sum_{k\in\omega} 2^{-k} < \infty. \ \square$$

The most interesting question raised by Theorems 7.4 and 7.5 is of course if the analogue of Conjecture 3.6 holds for $\mathsf{Smz}^\sharp$.

Question 7.9 (CH). Is it true that if the inclusion $\mathsf{Smz}^\sharp(\mathbb{G}) \subseteq \mathcal{M}^*(\mathbb{G})$ holds for a Polish group $\mathbb{G}$, then $\mathbb{G}$ is locally compact?

Another open problem is the rôle of TSI in Theorem 7.6.

Question 7.10. Can the TSI assumption in Theorem 7.6 be dropped, or weakened to CLI?

Continuous images and cartesian products of meager-additive sets. Since meager-additive sets coincide with $\mathsf{Smz}^\sharp$ sets in TSI locally compact groups, they are preserved by continuous mappings and by cartesian products.

Theorem 7.11. *Let $\mathbb{G}_1$ be a TSI Polish group and $\mathbb{G}_2$ a locally compact Polish group. Let $f : \mathbb{G}_1 \to \mathbb{G}_2$ a continuous mapping. If $X \subseteq \mathbb{G}_1$ is $\mathcal{M}$-additive, then so is $f(X)$.*

Theorem 7.12. *Let $\mathbb{G}_1, \mathbb{G}_2$ be TSI locally compact Polish groups. Let $X_1 \subseteq \mathbb{G}_1$, $X_2 \subseteq \mathbb{G}_2$.*

(i) *If X_1 is Smz and X_2 is $\mathcal{M}$-additive, then $X_1 \times X_2$ is Smz.*
(ii) *If X_1 and X_2 are $\mathcal{M}$-additive, then so is $X_1 \times X_2$.*

Corollary 7.13. *Let $\mathbb{G}_1, \mathbb{G}_2$ be TSI locally compact Polish groups. Let $X \subseteq \mathbb{G}_1\times\mathbb{G}_2$. The following are equivalent.*

(i) *X is $\mathcal{M}$-additive,*
(ii) *$\operatorname{proj}_1 X$ and $\operatorname{proj}_2 X$ are $\mathcal{M}$-additive,*
(iii) *$\operatorname{proj}_1 X \times \operatorname{proj}_2 X$ is $\mathcal{M}$-additive.*

We conclude this section with a few remarks on meager-additive sets in the Cantor set 2^ω. There are a few variations of $\mathcal{M}$-additivity: A set S in 2^ω is

- *$\mathcal{M}$-additive* if $\forall M \in \mathcal{M} \quad S + M \in \mathcal{M}$,
- *sharply $\mathcal{M}$-additive* if $\forall M \in \mathcal{M}\ \exists F \supseteq S\ F_\sigma \quad F + M \in \mathcal{M}$,
- *flatly $\mathcal{M}$-additive* if $\forall M \in \mathcal{M}\ \exists F \supseteq S\ F_\sigma \quad F + M \neq 2^\omega$,
- *$\mathcal{E}$-additive* if $\forall E \in \mathcal{E}\ S + E \in \mathcal{E}$,
- *sharply $\mathcal{E}$-additive* if $\forall E \in \mathcal{E}\ \exists F \supseteq S\ F_\sigma \quad F + E \in \mathcal{E}$.

The question whether $\mathcal{E}$-additive sets are related to $\mathcal{M}$-additive sets are related was posed by Nowik and Weiss [34]. Their question was answered in [54] by the following theorem. Let us note that while one would expect that, e.g., the proof of (ii)⇒(iv) is a matter of routine, it is actually surprisingly difficult.

Theorem 7.14 ([54]). *Let $S \subseteq 2^\omega$. The following properties of S are equivalent.*

(i) *S is* $\mathsf{Smz}^\sharp$*,*
(ii) *S is $\mathcal{M}$-additive,*
(iii) *S is sharply $\mathcal{M}$-additive,*
(iv) *S is flatly $\mathcal{M}$-additive,*
(v) *S is $\mathcal{E}$-additive,*
(vi) *S is sharply $\mathcal{E}$-additive.*

The definitions of these notions extend in a straightforward way to Polish groups (or in case $\mathcal{E}$ is considered, locally copact Polish groups). However, the proofs of the above theorem depends very much on the fine combinatorial structure of 2^ω and are thus not easily transferable to a context of a Polish group. With quite some effort the equivalence (i)⇔(ii)⇔(iii)⇔(iv) in TSI locally compact Polish groups was proved in [52]. (The equivalence (i)⇔(ii) is presented in this section as Theorem 7.6.) But the equivalences including $\mathcal{E}$ are still not understood.

Question 7.15 ([52])**.** Let $\mathbb{G}$ be a locally compact TSI Polish group.
(i) Is every $\mathcal{M}$-additive set in $\mathbb{G}$ $\mathcal{E}$-additive?
(ii) Is every $\mathcal{E}$-additive set in $\mathbb{G}$ $\mathcal{M}$-additive?
(iii) Is every $\mathcal{E}$-additive set in $\mathbb{G}$ sharply $\mathcal{E}$-additive?

8. Uniformity number of meager-additive sets

Since the notion of sharp measure zero is rather new, not much is known about the cardinal invariants of the σ-ideal of $\mathsf{Smz}^\sharp$ sets. We will investigate only the uniformity number of $\mathsf{Smz}^\sharp$ for metric spaces and Polish groups. We refer to section 4 for the notation.

Bartoszyński [1] and Pawlikowski [36] investigated and calculated a related cardinal – the uniformity number of the ideal of meager-additive sets in 2^ω. This cardinal invariant is termed *transitive additivity of* $\mathcal{M}$ and denoted by $\mathsf{add}^*(\mathcal{M})$. By [1, 2.7.14], $\mathsf{add}^*(\mathcal{M}) = \mathfrak{eq}^*$.

The two results on Polish groups, Theorem 8.3 and Corollary 8.4, come from [54]. The other results of this section are new.

Theorem 8.1. *Let X be a separable metric space that is not* $\mathsf{Smz}^\sharp$*.*
(i) $\mathsf{non}(\mathsf{Smz}^\sharp(\omega^\omega)) = \mathsf{add}(\mathcal{M})$*,*
(ii) $\mathsf{non}(\mathsf{Smz}^\sharp(2^\omega)) = \mathfrak{eq}^*$*,*
(iii) $\mathsf{add}(\mathcal{M}) \leqslant \mathsf{non}(\mathsf{Smz}^\sharp(X))$*,*
(iv) *if X is σ-totally bounded, then* $\mathfrak{eq}^* \leqslant \mathsf{non}(\mathsf{Smz}^\sharp(X))$*,*
(v) *if X is not of universal measure zero, then* $\mathsf{non}(\mathsf{Smz}^\sharp(X)) \leqslant \mathsf{non}(\mathcal{N})$*.*

Proof. (i) Recall that the metric on ω^ω is the least difference metric given by $d(f,g) = 2^{-n(f,g)}$, where $n(f,g) = \min\{n : f(n) \neq g(n)\}$.

Let $S \subseteq \omega^\omega$ be an unbounded set such that $|X| = \mathfrak{b}$. Since X is not bounded, it is not σ-totally bounded and in particular it is not $\mathsf{Smz}^\sharp$. It follows that $\mathsf{non}\,\mathsf{Smz}^\sharp(\omega^\omega) \leqslant \mathfrak{b}$.

Theorem 4.2(iii) yields a set $S \subseteq \omega^\omega$ that is not Smz and $|S| = \mathsf{cov}\,\mathcal{M}$. This set is clearly not $\mathsf{Smz}^\sharp$ and thus $\mathsf{non}\,\mathsf{Smz}^\sharp(\omega^\omega) \leqslant \mathsf{cov}\,\mathcal{M}$.

It follows that $\mathsf{non}\,\mathsf{Smz}^\sharp(\omega^\omega) \leqslant \min(\mathsf{cov}\,\mathcal{M}, \mathfrak{b})$. By a theorem of Miller [29] $\min(\mathsf{cov}\,\mathcal{M}, \mathfrak{b}) = \mathsf{add}(\mathcal{M})$ (see also [1, 2.2.9]). Thus we have $\mathsf{non}\,\mathsf{Smz}^\sharp(\omega^\omega) \leqslant \mathsf{add}(\mathcal{M})$.

In the other direction, suppose that $S \subseteq \omega^\omega$ is not $\mathsf{Smz}^\sharp$. By Theorem 7.5, S is not meager-additive set in the group $\mathbb{Z}^\omega$. Therefore there is a meager set $M \subseteq \mathbb{Z}^\omega$ such that $S + M = \bigcup_{s\in S}(M + s)$ is not meager and hence $|S| \geqslant \mathsf{add}(\mathcal{M})$. Thus $\mathsf{non}\,\mathsf{Smz}^\sharp(\omega^\omega) \geqslant \mathsf{add}(\mathcal{M})$.

(ii) By Theorem 7.4, a set $S \subseteq 2^\omega$ is $\mathsf{Smz}^\sharp$ if and only if it is meager-additive. Thus $\mathsf{non}\,\mathsf{Smz}^\sharp(2^\omega) = \mathsf{add}^*(\mathcal{M})$ and the latter equals by the aforementioned result of Bartoszyński to $\mathfrak{eq}^*$.

(iii) Let $\{z_m : m \in \omega\} \subseteq X$ be a dense set in X. To each $x \in X$ assign a function $\widehat{x} \in \omega^\omega$ defined by

$$\widehat{x}(n) = \min\{m : d(x, z_m) < 2^{-n}\}. \tag{12}$$

We claim that the inverse map $\widehat{x} \mapsto x$ is Lipschitz. Indeed, if $d(\widehat{x}, \widehat{y}) = 2^{-n}$, then $\widehat{x}(n-1) = \widehat{y}(n-1)$ and in particular $\exists m\ d(x, z_m) < 2^{-n+1}$ and $d(y, z_m) < 2^{-n+1}$. Therefore $d(x, y) < 2 \cdot 2^{-n+1}$, whence $d(x, y) < 4d(\widehat{x}, \widehat{y})$. In particular $\widehat{x} \mapsto x$ is uniformly continuous. So if $S \subseteq X$ is not $\mathsf{Smz}^\sharp$, then $\widehat{S} = \{\widehat{x} : x \in S\}$ is not $\mathsf{Smz}^\sharp$ as well. If follows that $\mathsf{non}(\mathsf{Smz}^\sharp(X)) \geqslant \mathsf{non}(\mathsf{Smz}^\sharp(\omega^\omega))$ and (iii) follows from (i).

(iv) Consider the completion X^* of X. Since X is σ-totally bounded, there is a σ-compact set $K \subseteq X^*$ that contains X. Let $K_n \nearrow K$ be a sequence of compact sets. Suppose that $S \subseteq X$ is a not $\mathsf{Smz}^\sharp$ set such that $|S| = \mathsf{non}(\mathsf{Smz}^\sharp(X))$. There is n such that $S \cap K_n$ is not $\mathsf{Smz}^\sharp$, therefore $|S \cap K_n| = \mathsf{non}(\mathsf{Smz}^\sharp(X))$. Since K_n is compact, it is *dyadic*: there is a uniformly continuous mapping $f : 2^\omega \to K_n$ onto K_n. For each $x \in S \cap K_n$ pick $\widehat{x} \in f^{-1}(x)$ and set $\widehat{S} = \{\widehat{x} : x \in S \cap K_n\}$. Then $f(\widehat{S}) = S \cap K_n$, hence $\widehat{S}$ is not $\mathsf{Smz}^\sharp$, and clearly $|\widehat{S}| = |S \cap K_n| = \mathsf{non}(\mathsf{Smz}^\sharp(X))$. It follows that $\mathsf{non}(\mathsf{Smz}^\sharp(2^\omega)) \leqslant \mathsf{non}(\mathsf{Smz}^\sharp(X))$ whence $\mathsf{add}^*(\mathcal{M}) \leqslant \mathsf{non}(\mathsf{Smz}^\sharp(X))$ by (ii).

(v) is a trivial consequence of Theorem 4.2(ii). □

As expected, for analytic metric spaces we can do better. Recall that a metric space has the *small ball property* if it admits a base $\{B_n\}$ such that $\operatorname{diam} B_n \to 0$. This notion is due to Behrends and Kadec [3]. We refer to [19] for more information.

Theorem 8.2. *Let X be an uncountable analytic metric space.*

(i) $\mathsf{add}(\mathcal{M}) \leqslant \mathsf{non}(\mathsf{Smz}^\sharp(X)) \leqslant \mathfrak{eq}^*$,
(ii) *if X is σ-totally bounded, then* $\mathsf{non}(\mathsf{Smz}^\sharp(X)) = \mathfrak{eq}^*$,
(iii) *if X does not have the small ball property, then* $\mathsf{non}(\mathsf{Smz}^\sharp(X)) = \mathsf{add}(\mathcal{M})$.

Proof. (i) The left-hand inequality is Theorem 8.1(iii). Since X contains a (uniform) copy of 2^ω, Theorem 8.1(ii) yields $\mathfrak{eq}^* = \mathsf{non}(\mathsf{Smz}^\sharp(2^\omega)) \geqslant \mathsf{non}\,\mathsf{Smz}^\sharp(\mathbb{G})$.

(ii) follows from (i) and Theorem 8.1(iv).

(iii) Consider the mapping $x \mapsto \widehat{x}$ defined by (12). Let $B \subseteq \omega^\omega$ be an unbounded set such that $|B| = \mathfrak{b}$. As shown in [19, 4.4], the set $\widehat{X} = \{\widehat{x} : x \in X\}$ is dominating in ω^ω. Therefore for each $f \in B$ there is $x_f \in X$ such that $\widehat{x}_f \geqslant^* f$. Set $S = \{x_g : f \in B\}$. Then $\widehat{S}$ is not bounded, because it dominates B. It follows that S is not σ-totally bounded in X, and in particular it is not $\mathsf{Smz}^\sharp$. Since clearly $|\widehat{S}| = |B| = \mathfrak{b}$, we conclude that $\mathsf{non}(\mathsf{Smz}^\sharp(X)) \leqslant \mathfrak{b}$. By (i) also $\mathsf{non}(\mathsf{Smz}^\sharp(X)) \leqslant \mathsf{add}(\mathcal{M})$, so $\mathsf{non}(\mathsf{Smz}^\sharp(X)) \leqslant \min(\mathfrak{eq}^*, \mathfrak{b}) = \mathsf{add}(\mathcal{M})$, and the reverse inequality follows from (i). □

We will now calculate uniformity numbers of $\mathsf{Smz}^\sharp(\mathbb{G})$ for CLI Polish groups and of $\mathcal{M}^*(\mathbb{G})$ for TSI Polish groups.

Theorem 8.3 ([54])**.** *Let $\mathbb{G}$ be a CLI Polish group.*

(i) *If $\mathbb{G}$ is locally compact, then $\mathsf{non}(\mathsf{Smz}^\sharp(\mathbb{G})) = \mathfrak{eq}^*$.*
(ii) *If $\mathbb{G}$ is not locally compact, then $\mathsf{non}(\mathsf{Smz}^\sharp(\mathbb{G})) = \mathsf{add}(\mathcal{M})$.*

Proof. (i) is a straightforward from Theorem 8.2(ii).

(ii) By Lemma 4.3, $\mathbb{G}$ contains a uniform copy of ω^ω. Therefore $\mathsf{non}(\mathsf{Smz}^\sharp(\mathbb{G})) \leqslant \mathsf{non}(\mathsf{Smz}^\sharp(\omega^\omega))$. Now apply Theorem 8.1(i) to get $\mathsf{non}(\mathsf{Smz}^\sharp(\mathbb{G})) \leqslant \mathsf{non}(\mathcal{M})$. The reverse inequality is straightforward from Theorem 8.2(i). □

The following is a trivial consequence of this theorem and Theorem 7.4.

Corollary 8.4 ([54])**.** *Let $\mathbb{G}$ be a TSI Polish group.*

(i) *If $\mathbb{G}$ is locally compact, then $\mathcal{M}^*(\mathbb{G}) = \mathfrak{eq}^*$.*
(ii) *If $\mathbb{G}$ is not locally compact, then $\mathcal{M}^*(\mathbb{G}) = \mathsf{add}(\mathcal{M})$.*

We conclude with a simple argument that shows that $\mathsf{Smz}^\sharp$ is consistently not a topological property. The Baer-Specker group $\mathbb{Z}^\omega$ is a TSI Polish group. On the other hand, it is homeomorphic to the set of irrational numbers, so regard it as a subset of $\mathbb{R}$. By Theorem 8.3(ii) there is a set $X \subseteq \mathbb{Z}^\omega$ such that $|X| = \mathsf{add}(\mathcal{M})$ and that is not $\mathsf{Smz}^\sharp$ in the invariant metric. On the other hand, if $\mathsf{add}(\mathcal{M}) < \mathsf{add}^* \mathcal{M}$, then X is by Theorem 8.3(i) $\mathsf{Smz}^\sharp$ in the metric of the real line. Since $\mathsf{add}(\mathcal{M}) < \mathsf{add}^* \mathcal{M}$ is relatively consistent with ZFC (as proved by Pawlikowski [36]), X is consistently a set that is $\mathsf{Smz}^\sharp$ is one metric on $\mathbb{Z}^\omega$ and not $\mathsf{Smz}^\sharp$ in another homeomorphic metric.

References

1. Tomek Bartoszyński and Haim Judah, *Set theory*, A K Peters Ltd., Wellesley, MA, 1995, On the structure of the real line. MR 1350295 (96k:03002)
2. James E. Baumgartner, *Iterated forcing*, Surveys in set theory, London Math. Soc. Lecture Note Ser., vol. 87, Cambridge Univ. Press, Cambridge, 1983, pp. 1–59. MR 823775
3. Ehrhard Behrends and Vladimir M. Kadets, *Metric spaces with the small ball property*, Studia Math. **148** (2001), no. 3, 275–287. MR 1880727 (2003b:54032)
4. A. S. Besicovitch, *Concentrated and rarified sets of points*, Acta Math. **62** (1933), no. 1, 289–300. MR 1555386
5. ———, *Correction*, Acta Math. **62** (1933), no. 1, 317–318. MR 1555389
6. E. Borel, *Sur la classification des ensembles de mesure nulle*, Bull. Soc. Math. France **47** (1919), 97–125. MR 1504785
7. Miguel A. Cardona, *Yorioka's characterization of the cofinality of the strong measure zero ideal and its independency from the continuum*, Submitted April 2019.
8. Miguel A. Cardona, Diego A. Mejía, and Ismael E. Rivera-Madrid, *The covering number of the strong measure zero ideal can be above almost everything else*, Submitted on 5 Feb 2019.
9. Timothy J. Carlson, *Strong measure zero and strongly meager sets*, Proc. Amer. Math. Soc. **118** (1993), no. 2, 577–586. MR 1139474 (94b:03086)
10. Paul Corazza, *The generalized Borel conjecture and strongly proper orders*, Trans. Amer. Math. Soc. **316** (1989), no. 1, 115–140. MR 982239 (90c:03042)
11. Tadeusz Dobrowolski and Witold Marciszewski, *Covering a Polish group by translates of a nowhere dense set*, Topology Appl. **155** (2008), no. 11, 1221–1226. MR 2421832 (2009d:03112)
12. D. H. Fremlin, *Measure theory. Vol. 5, set-theoretic measure theory*, Torres Fremlin, Colchester, 2015.
13. Fred Galvin, Jan Mycielski, and Robert M. Solovay, *Strong measure zero sets*, Notices of the American Mathematical Society **26** (1979), no. 3, A–280.

14. ______, *Strong measure zero and infinite games*, Arch. Math. Logic **56** (2017), no. 7-8, 725–732. MR 3696064
15. Martin Goldstern, Haim Judah, and Saharon Shelah, *Strong measure zero sets without Cohen reals*, J. Symbolic Logic **58** (1993), no. 4, 1323–1341. MR 1253925 (95c:03126)
16. Alfred Haar, *Der Massbegriff in der Theorie der kontinuierlichen Gruppen*, Ann. of Math. (2) **34** (1933), no. 1, 147–169. MR 1503103
17. J. D. Howroyd, *On Hausdorff and packing dimension of product spaces*, Math. Proc. Cambridge Philos. Soc. **119** (1996), no. 4, 715–727. MR 1362951 (96j:28006)
18. Michael Hrušák, David Meza-Alcántara, and Hiroaki Minami, *Pair-splitting, pair-reaping and cardinal invariants of F_σ-ideals*, J. Symbolic Logic **75** (2010), no. 2, 661–677. MR 2648159 (2012e:03099)
19. Michael Hrušák, Wolfgang Wohofsky, and Ondřej Zindulka, *Strong measure zero in separable metric spaces and Polish groups*, Arch. Math. Logic **55** (2016), no. 1-2, 105–131. MR 3453581
20. Michael Hrušák and Jindřich Zapletal, *Strong measure zero sets in Polish groups*, Illinois J. Math. **60** (2016), no. 3-4, 751–760. MR 3707641
21. Thomas Jech, *Set theory*, Springer Monographs in Mathematics, Springer-Verlag, Berlin, 2003, The third millennium edition, revised and expanded. MR 1940513 (2004g:03071)
22. Alexander S. Kechris, *Classical descriptive set theory*, Graduate Texts in Mathematics, vol. 156, Springer-Verlag, New York, 1995. MR 1321597 (96e:03057)
23. J. D. Kelly, *A method for constructing measures appropriate for the study of Cartesian products*, Proc. London Math. Soc. (3) **26** (1973), 521–546. MR 0318427
24. Ljubiša D. R. Kočinac and Marion Scheepers, *Combinatorics of open covers. VII. Groupability*, Fund. Math. **179** (2003), no. 2, 131–155. MR 2029229
25. Kenneth Kunen, *Set theory*, Studies in Logic and the Foundations of Mathematics, vol. 102, North-Holland Publishing Co., Amsterdam, 1983, An introduction to independence proofs, Reprint of the 1980 original. MR 756630 (85e:03003)
26. Marcin Kysiak, *On Erdős-Sierpiński duality between Lebesgue measure and Baire category (in Polish)*, Master's thesis (2000).
27. Paul B. Larson, *The stationary tower*, University Lecture Series, vol. 32, American Mathematical Society, Providence, RI, 2004, Notes on a course by W. Hugh Woodin. MR 2069032
28. Richard Laver, *On the consistency of Borel's conjecture*, Acta Math. **137** (1976), no. 3-4, 151–169. MR 0422027 (54 #10019)
29. Arnold W. Miller, *Some properties of measure and category*, Trans. Amer. Math. Soc. **266** (1981), no. 1, 93–114. MR 613787 (84e:03058a)
30. Arnold W. Miller and David H. Fremlin, *On some properties of Hurewicz, Menger, and Rothberger*, Fund. Math. **129** (1988), no. 1, 17–33. MR 954892 (89g:54061)
31. Arnold W. Miller and Juris Steprāns, *The number of translates of a closed nowhere dense set required to cover a Polish group*, Ann. Pure Appl. Logic **140** (2006), no. 1-3, 52–59. MR 2224048 (2007a:03059)
32. M. E. Munroe, *Introduction to measure and integration*, Addison-Wesley Publishing Company, Inc., Cambridge, Mass., 1953. MR 0053186 (14,734a)
33. Andrej Nowik, Marion Scheepers, and Tomasz Weiss, *The algebraic sum of sets of real numbers with strong measure zero sets*, J. Symbolic Logic **63** (1998), no. 1, 301–324. MR 1610427 (99c:54049)
34. Andrzej Nowik and Tomasz Weiss, *On the Ramseyan properties of some special subsets of 2^ω and their algebraic sums*, J. Symbolic Logic **67** (2002), no. 2, 547–556. MR 1905154 (2003c:03087)
35. Kalyanapuram Rangachari Parthasarathy, *Introduction to probability and measure*, Springer-Verlag New York Inc., New York, 1978. MR 0651013
36. Janusz Pawlikowski, *Powers of transitive bases of measure and category*, Proc. Amer. Math. Soc. **93** (1985), no. 4, 719–729. MR 776210 (86k:03041)
37. ______, *Finite support iteration and strong measure zero sets*, J. Symbolic Logic **55** (1990), no. 2, 674–677. MR 1056381
38. Karel Prikry, unpublished result.
39. C. A. Rogers, *Hausdorff measures*, Cambridge University Press, London, 1970. MR 0281862 (43 #7576)

40. Fritz Rothberger, *Sur les familles indénombrables de suites de nombres naturels et les problèmes concernant la propriété C*, Proc. Cambridge Philos. Soc. **37** (1941), 109–126. MR 0004281 (2,352a)
41. Marion Scheepers, *Finite powers of strong measure zero sets*, J. Symbolic Logic **64** (1999), no. 3, 1295–1306. MR 1779763 (2001m:03096)
42. Saharon Shelah, *Every null-additive set is meager-additive*, Israel J. Math. **89** (1995), no. 1-3, 357–376. MR 1324470 (96j:04002)
43. W. Sierpiński and E. Szpilrajn, *Remarque sur le problème de la mesure*, Fund. MAth. **26** (1936), 256–261.
44. Wacław Sierpiński, *Sur un ensemble non denombrable, dont toute image continue est de mesure nulle*, Fundamenta Mathematicae **11** (1928), 302–304.
45. Edward Szpilrajn, *Remarques sur les fonctions sousharmoniques*, Ann. of Math. (2) **34** (1933), no. 3, 588–594. MR 1503125
46. ———, *Remarques sur les fonctions complètement additives d'ensemble et sur les ensembles jouissant de la propriété de Baire*, Fund. Math. **22** (1934), no. 1, 303–311.
47. Tomasz Weiss, *On meager additive and null additive sets in the Cantor space* 2^ω *and in* $\mathbb{R}$, Bull. Pol. Acad. Sci. Math. **57** (2009), no. 2, 91–99. MR 2545840
48. ———, *Addendum to "On meager additive and null additive sets in the Cantor space* 2^ω *and in* $\mathbb{R}$*" (Bull. Polish Acad. Sci. Math. 57 (2009), 91–99) [mr2545840]*, Bull. Pol. Acad. Sci. Math. **62** (2014), no. 1, 1–9. MR 3241126
49. ———, *Properties of the intersection ideal* $\mathcal{M} \cap \mathcal{N}$ *revisited*, Bull. Pol. Acad. Sci. Math. **65** (2017), no. 2, 107–111. MR 3731016
50. Wolfgang Wohofsky, *Special sets of real numbers and variants of the Borel Conjecture*, Ph.D. thesis, Technische Universität Wien, 2013.
51. Teruyuki Yorioka, *The cofinality of the strong measure zero ideal*, J. Symbolic Logic **67** (2002), no. 4, 1373–1384. MR 1955243 (2004b:03071)
52. Ondřej Zindulka, *Meager-additive sets in topological groups*, Submitted.
53. ———, *Universal measure zero, large Hausdorff dimension, and nearly Lipschitz maps*, Fund. Math. **218** (2012), no. 2, 95–119. MR 2957686
54. ———, *Strong measure zero and meager-additive sets through the prism of fractal measures*, Comment. Math. Univ. Carol. **60** (2019), no. 1, 131–155.

Centro de Ciencias Matemáticas, Universidad Nacional Autónoma de México, Cámpus Morelia, 58089, Morelia, Michoacán, México.
Email address: michael@matmor.unam.mx
URL: http://www.matmor.unam.mx/~michael

Department of Mathematics, Faculty of Civil Engineering, Czech Technical University, Thákurova 7, 160 00 Prague 6, Czech Republic
Email address: ondrej.zindulka@cvut.cz
URL: http://mat.fsv.cvut.cz/zindulka